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Investigating learning mathematics using dynamic software: A case study of understanding the concept of slope

MA in Mathematics Education

Institute of Education
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Abstract

DGS based activities which focus on multiple representations of functions and making connection among different areas of mathematics could help students to gain insight in the mathematical concept and provide a useful environment for developing their conceptualizations. Dynamic environments could influence students understanding in various ways. This study attempts to investigate students` learning linear equation and particularly the slope of a line and its y-intercept based on proposed theoretical framework and through Dynamic Geometry Software activities. It provides a description of the change as a result of students` interactions with the dynamic environment and it also describes, in detail, how the students acquired a new point of view about what they had already taught. Five students who studied in ۸th and ۹th grade were initially interviewed in order to show their understanding of the concept of slope and y-intercept. Then they were asked to work with dynamic based activities and their interactions were observed. Finally, students were interviewed and responded to three initial interview questions again in order for the researcher to identify a possible improvement and change of understanding linear equations.

After careful observation of working sessions, audio-recording and transcribing all interviews and preparing observation schedule a qualitative method were used to analyse the data collected for this study. The data were analysed in three levels of knowledge and students' solutions and responses were interpreted based on two dimensional framework. There are two ways of viewing functions (the process and object perspectives) and the three representations of functions (tabular, graphical, and algebraic). Findings demonstrated that students learnt to have a graphical interpretation of the slope and use the graphical method in addition to algebraic method sketch graph of lines or write an equation for given line. Analysing data indicated that, appropriate computer based activities enhanced students to move fluently across different representations and perspectives, to be able to see the slope of line and y–intercept as object, although developing such flexibility needs more research. Research should be planned in order to complete the framework and appropriate activities should be designed to show students` movement from one cell of the framework to another or across different representations in a perspective or from one perspective to another one.

١- Introduction

- - *Preface*

In my country, Iran, dynamic geometry was unknown for most of mathematics teachers till my colleague and I provided a rich report to introduce dynamic geometry and dynamic environments for the Organization for Educational Research and Development in the Ministry of Education, in ١٩٩٩. Since then, educators, curriculum developer, and teachers became more familiar with dynamic geometry and efforts were done to develop using its applications in the related area. For example, an online course was designed for mathematics teachers in order to make them familiar with particular dynamic geometry software, Geogebra. Also, more than five hundred activities for variety of mathematics topics in different grades were designed and presented on a Web-site. At the same time, I used some of these activities for teaching students in grade eight and observed the influence of using dynamic software on students` learning and their understanding mathematics.

During that period of time, I was involved in revising middle school mathematics textbooks and developing new national mathematics curriculum in Ministry of Education. One of the considered areas in the new curriculum was using digital technology in mathematics education. My interest in choosing this topic for dissertation was initiated by the need to take ‘new’ ideas to influence my career as a mathematics curriculum developer and teacher. Therefore, I hope that my study could help curriculum developers to rethink the impact of technology on designing mathematic curriculum and textbooks in Iran and assist teachers to develop their understanding to use dynamic software in the process of learning-teaching mathematics.

The mathematics topic involved in this study is the linear equation and particularly the slope of a line and its y-intercept. This topic is presented in grade eight in educational system in Iran and students have difficulties to understand it adequately and the result of TIMSS (١٩٩٥, ١٩٩٩, ٢٠٠٣ and ٢٠٠٧) clearly indicated these difficulties and supported my experience as a mathematics teacher and curriculum developer. In addition to above mentioned reason, there are two more reasons for this topic being chosen and implemented using Geogebra. First, it has the potential to make connection between geometric and algebraic aspects of this topic. Second, making this connection helps students to move fluently through variety of representations of linear equation in order to understand the concept of slope.

- - *Rationale*

The algebra standard for Grade ٦-٨ in the NCTM principles and standard for school mathematics (٢٠٠٠) includes the following statements:

In grades ٦–٨ all students should:

- relate and compare different forms of representation for a relationship;
- identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations;
- explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope;
- use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships;
- recognize and generate equivalent forms for simple algebraic expressions and solve linear equations;
- model and solve contextualized problems using various representations, such as graphs, tables, and equations;
- use graphs to analyse the nature of changes in quantities in linear relationships.

Also, the algebra standards for Grades ٩–١٢ extend expectation of students in the following statements:

In grades ٩–١٢ all students should:

- understand relations and functions and select, convert flexibly among, and use various representations for them;
- generalise patterns using explicitly defined and recursively defined functions;
- analyse functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behaviour;
- use symbolic algebra to represent and explain mathematical relationships;
- use a variety of symbolic representations, including recursive and parametric equations, for functions and relations;
- judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.

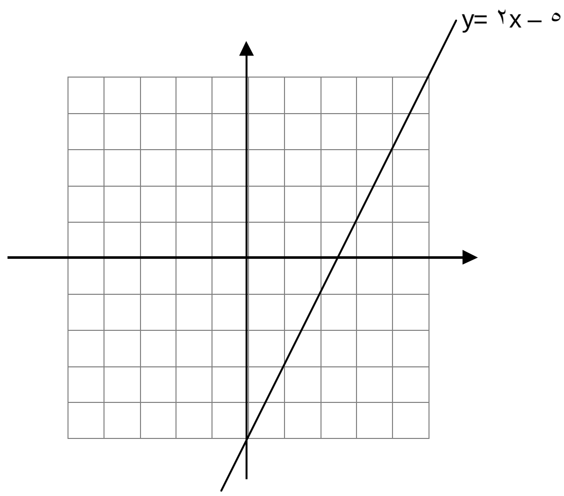
The linear equation is presented in grade eight in mathematics textbooks in Iran. This topic seems difficult for students and when the mathematics became complicated, their performance fell off dramatically. My experience as a teacher indicates that the linear equation is the most difficult topic in grade eight and the result of local and national exams in this grade point out this difficulty. Researches show same problem in other countries too (TIMSS, ٢٠٠٣ and ٢٠٠٧; Moschkovich et al., ١٩٩٣). For example, only ٣١% of the Iranian students participated in TIMSS ٢٠٠٧ answered the following question, while this type of question exists in mathematics textbook.

The table (below) shows the relation between two variables x and y.

x	1	2	3	4	5
y	1	2	5	7	9

Which equation shows this relation?

Also, Moschkovich et al. (١٩٩٣) cites from the National Assessment of Educational Progress and expresses that high school pupils had considerable difficulty with the following question:



a) On the axes above, draw a line parallel to y= ٢x – ٥ that goes through the origin ٠.

b) On the line below, write an equation of the new line

Equation: --

> Only ٣٢% of the high school seniors drew the new parallel line on the graph, when a correct response essentially required the ability to find the origin O on the graph, the

> ability to find the exiting line on the graph, and an understanding of the term "parallel". Sixteen percent of the twelfth graders answered both parts of this question correctly.

(Mullis et al., ١٩٩١, p. ١١ as cited in Moschkovich et al., ١٩٩٣, p. ٧٠)

This question can become more difficult and complex if in its statement 'the origin' is replaced by any other point in the plane (e. g. point (١, ٤)). The required knowledge to answer this question is that the desired line has the same slope as the given line, deducing that its slope is ٢; and it passes through the origin, so its y intercept is ٠, therefore its equation is y=٢x. The difficulty of this question stems from the lake of understanding the relationship between different representations of the concept of function. Students must come to understand connections across the meanings of algebraic representation in a geometric context with different perspectives regarding the functions (Moschkovich et al., ١٩٩٣).

Those perspectives are the process perspective and the object perspective. In the process perspective, a function is perceived as correspondence between x and y values, For example, in above question, the origin is a point of the required line if and only if its coordinates satisfy the equation of line (y= mx + n). Hence, ٠= m (٠) + n, and n=٠. From the object perspective, a function and any of its representations (algebraic or as graph in the plane) are conceptualized as entities. The slope of a line is one attribute of the line as an encapsulated concept and therefore in order to determine the equation of line, it is necessary to conceive m as an object (Moschkovich et al., ١٩٩٣). Since parallel lines have the same slope, one can recognise it directly from its equation, as the coefficient of x. Therefore m = ٢, and the equation of the required line is y=٢x + ٠.

Several researches emphasize on the object and the process perspectives as an important part of learning functions and graphs (see Sfard, ١٩٩١; Moschkovich et al., ١٩٩٣; Kieran, ١٩٩٣). This study attempts to investigate that which perspective and representation can be beneficially used in particular problem contexts by students, and how dynamic computer environment can help them to move flexibly among them to interpret and solve problems in this domain. One aspect of understanding linear equation is the capability to move fluently between two perspectives (process and object) in a variety of representations (tabular, algebraic and graphical).

Sfard (١٩٩١) through analysing variety of mathematical definitions and representations presents a theoretical framework about the role of algorithms in mathematical thinking and concludes that abstract notions, such as function, can be understood in two complementary ways: as objects (structurally), and as processes (operationally). A mathematical entity as an object is a real thing that exist somewhere in space and time and can be manipulated as a whole, without going into details. The first step in the acquisition of a new mathematical concept, often, is the operational conception; however shifting from computational operations to abstract objects is intrinsically difficult process and takes long time. The next chapter of this dissertation briefly provides background of related research in order to elaborate a framework for this study.

The elaboration of the process and object perspective in the case of linear relations points to some conceptual difficulty to understand the slope of a line. Considering these two perspectives, I adapt a ٢ by ٣ matrix (framework) based on a study of Moschkovich et al. (١٩٩٣) and illustrate how this framework can be used to interpret collected data. Moschkovich et al. (١٩٩٣) expanded this framework as a guide for interpreting and understanding students` solution to problem in the domain of function and particularly linear equation. The following quote indicate their claim:

> Our analysis of this domain indicate that such competence rests in part on the ability to know which representations and perspectives are likely to be useful in particular problem contexts, and to switch flexibly among representations and perspectives as seems appropriate and can high light the kinds of connections one needs to make and also can serve as a guide for interpreting and understanding students` solutions to problem in this domain (p. ٧٤)

The concept of function is a central in mathematics and important part of the current curriculum in high school in Iran. Although there are several ways of presenting functions, but normally functions are represented algebraically at the first step and students learn to manipulate functions by manipulating the symbols. Then functions are represented graphically and students can also learn to manipulate functions by manipulating the graphs. Then students are expected to find connection between these two representations. For example, in grade ٩ in Iran, students firstly manipulate algebraic representation of a parabola and then the graph of parabola is briefly introduced without a sensible connection between these two representations. While Yerushalmy and Schwartz (١٩٩٣) argue that allowing students to use a rich set of both symbolic and graphic representations builds a deeper and richer understanding of the mathematical concepts such as function. They adopt Tall`s (١٩٨٩) categories on a variety of ‘cognitive obstacles’ in the learning of algebra, in order to discuss the source of the difficulties students encounter in learning about functions and their graphs and point out these obstacles:

- Algebraic situations normally are introduced in sensible environments and students pay more attention to algebraic representation rather than graph representation.
- The learning of graphs of functions usually is introduced as a final step and normally occurs after a long period of manipulating numerically and symbolically.
- Each representation presents in a separate system with no constructive interaction for the learner.

In this study, I investigate students` learning linear equation based on proposed theoretical framework through activities using dynamic software. I seek to gain insight to students` interactions through these activities and tasks, which leads to design appropriate activities in dynamic environment and to find different aspects of students` understanding of a particular topic of mathematics.

٢ - ٣ - *Dynamic geometry software*

Dynamic geometry software enables users to simultaneously manipulate the graphical representations and observe its impacts on the tabular or symbolic representations as a result of this manipulation. The multiple related representations in dynamic software environment provide an opportunity for direct manipulation of both symbolic and graphic representations. The essential features of these kinds of software environments are that: (١) Various representations are related; (٢) Manipulating each representation can change the structure of both representations. Romberg et al. (١٩٩٣, p ix) draws on computers` role in developing understanding the concept of function and its graphical representation and states: “With the increasing use of computers and graphing calculators, a coherent body of knowledge about how the connections are developed among tables, graphs, and the algebraic expressions related to functions is desperately needed.”

١-٣-١- Geogebra

Dynamic Geometry Systems (DGS) such as Cabri and Cinderella, and Computer Algebra Systems (CAS) such as Mathematica, Maple and Derive have widely influenced both on learning -teaching process and mathematics education research. Any of these two types of tools are used in separate domains. GeoGebra is a software that joins both domains (Hohenwatrter, ٢٠٠٤). A bidirectional combination of dynamic geometry and computer algebra systems had been already introduced by Schumann and Green (٢٠٠٠). The following quotation indicates their idea: “The three solution protocols [graphical, numerical, and algebraic] should not be considered separate, but rather as constituting a holistic comprehensive computer-aided approach” (p. ٣٢٤).

In ٢٠٠١, Markus Hohenwatrter began to develop a completely new kind of tool for mathematics education (GeoGebra - a Software System for Dynamic Geometry and Algebra in the Plane) as his master’s dissertation and later continued as part of his PhD thesis. GeoGebra is an interactive software offers algebraic manipulating such as entering equations directly using symbolic representations and encourages students to discover mathematics in a bidirectional experimental way (Hohenwatrter, ٢٠٠٤). The basic objects in GeoGebra are points, vectors, segments, polygons, straight lines and functions. Furthermore, users can directly input coordinates of points or vectors, equations of lines, functions and numbers or angles.

- - *The slope of a line*

Reviewing related literature gives an idea that the use of DGS is not as straightforward as originally thought. The readings offered me a framework of learning functions and particularly linear equations using DGS which highlights the main focus of literature as well as pointing out a ‘gap’ for further research. Furthermore reading about claims and linked research gives a broader perspective on the area of interest. Subsequently, I realised that first of all, it is important to understand what ‘straight lines’ means. Implicitly, it is important for one to know what should be taught and how it should be learnt. This is a requirement for designing activities and tasks in any dynamic environment. Schoenfeld (١٩٩٠, p. ٢٨٢-٢٨٣) truly illustrated these aspects of understanding of the line and its graph in eight parts. This category was a nice start for focusing on a particular topic in mathematics.

١. “General geometric knowledge about lines”. It includes understanding of concepts such as point, angle, line, parallel and also knowing that two points or a point and the direction determine a line.

٢. “Knowledge of algebra”. It includes equation solving, functions and how the lines are expressed algebraically.

٣. “Knowledge about the Cartesian plane”: Knowing how to plot points and drawing a graph such as lines.

٤. “Knowledge of the relationships between functions and their graphs”. For instance, understanding that when a point satisfies a linear equation.

٥. “Knowledge of algebraic conventions regarding lines and their entailments in the Cartesian plane”. This part is exactly related to my study and includes the concept of slope and y- intercept in the standard form of linear equation (y= mx + n). In this form m is the slope of the line and n is its y- intercept which means that the line crosses the y-axis at the point (٠, n). Different values of slope (m) effect on graph`s orientations.

٦. “Thinking simultaneously of the graph y = mx +n as a collection of points” and “as a single entity”. Different values of m and n change graphical entailments. For example if the value of n increases, the graph of a line comes down in the Cartesian plane.

٧. “Knowing different algebraic representations for straight lines”. Knowing that there are different forms to characterize a line (the two – intercept, the two – point, the point – slope) and they are all related to the same line, but in different circumstances.

٨. “Knowing all of the above in such a way that knowledge of one aspect of straight lines constrains knowledge in any of the others”.

This study focuses on the changes in students’ mathematical knowledge and understanding of already learned concept of slope of a line and its y-intercept in the Cartesian plane. It provides a description of the change as a result of their interactions with the dynamic environment and it also describes, in detail, how the students acquired a new point of view about what they had already taught. An overall view of the cognitive framework for the concept of slope that I adapted in my study is based on Schoenfeld and et al’s (١٩٩٣) presented four levels of knowledge. As they stated in their study, these levels are “present in any knowledge organization” and this structure is “compatible with the literature in cognitive psychology” (P. ٥٦). I illustrate these levels with brief explanation in order to define a framework for analysis of data in my study.

Level ١; macro organization: this level is reflection of our knowledge and perception based on the general observation at the macro level. For example, “y= mx + n” is a standard equation of a line, where m is slope and n is its y – intercept. Level ٢; Concepts and their associated properties: Level ٢ describes objects in the domain and their familiar (relevant) properties without formal justification. For instance, a positive slope describes that line slants down to the right and the larger the magnitude of slope is the line is closer to vertical one. Lines with negative slope slant down ward to the right, and lines of small slope are nearly horizontal. Also the point (٠, n) is called the y – intercept. Level ٣; Fine grained knowledge structure (primitive elements and connection among them). The connection illustrated at this level is the foundation on which competent performance rests. For the slope of a line, it is a link between the manipulations in the algebraic world namely “m = (y٢ – y١) / (x٢ – x١)” and the geometric world namely graphs. Also, if x = ٠, then the point (٠, n) is on both the line and the y – axis. This level is called, the Cartesian connection. Level ٤; contextual (primitive elements are indexed by contexts.)

In this study, the following Schoenfeld, et al’s (١٩٩٣) elements of concept of slope of a line are discussed in first three above mentioned levels: (١) two slot schema (slope, y–intercept), (٢) nature of m (computing m from coordinates of two points), (٣) sign of m (٤) magnitude of m, (٥) slope – intercept independence.

- *Aim of the study*

This study focuses on lines and graphs for several reasons. First, lines and graphs represent the first topics in mathematics textbooks in Iran at which pupils use symbolic system (linear equation or data patterns) to expand and understand their graphs. Second, graphing can be seen as one of the critical elements in early mathematics (Leinhardt et al., ١٩٩٠). Third, computers can provide a useful opportunity for students to see the connection between different representations of a line. One of the features that encouraged me to do this study is that algebraic and graphical representations which construct and define the concept of function are related to two different systems (Leinhardt et al., ١٩٩٠).

The main purpose of this study is to investigate the influence of deploying a dynamic geometry software package, Geogebra, on students` learning and understanding mathematics. The mathematics topics involved in this study is linear equation and it is chosen from mathematics textbooks in grade eight. More specifically, this study attempts to observe and investigate students' abilities and understanding linear equation and particularly the slope of lines through working on activities designed in dynamic geometry environment.

The key question I want to explore here is that how can these activities change learners' understanding of the slope of a line and its y-intercept. In this study I ask some pupils to work on activities which are designed and I want to compare their understanding before and after working with this software and observe the change in their learning during involvement in these activities. I interview pupils to explore how these activities help them to change their abilities and knowledge about linear equations, and follow the stories of this change to show a possible improvement.

١-٥-١- Hypothesis and Questions

The theoretical perspective of the present study is based on the assumption that DGS based activities which focus on the multiple connections and multiple representations of lines can help children to abstract the concept of slope and provide a useful environment for developing their conceptualizations. It also is assumed that emphasizing graphical representations will make functions and particularly linear equation easier to learn and can be used for most students, but it is not clear at what point and in what ways graphing software should be used as a teaching tool. Thus, I attempt to find appropriate answer for these questions:

١- In which ways do students think about different aspects of the concept of slope of line and its y-intercept?

٢- How using activities based on dynamic geometry software can change pupils` understanding of the slope of line?

٣- How designing appropriate tasks influences students` understanding to move fluently through variety of representations of linear equation?

٢- Literature review

- - *Preface*

Generally, books, journal research articles, curriculum documentations, textbooks and reports (governmental and international such as TIMSS) give a broad perspective on the area of this study. The related area consist three main domains in mathematics education. First, the outline of suggested definitions and perspectives of function will be reviewed. Second, looking at different theories of learning and understanding, the nature of mathematics in general and function (linear equation) in particular is important to establish a theoretical framework to explain the essential role which multiple representations have played in the process of teaching and learning. Third, documentation about this learning taking place within a computer environment and the use of computer technology in general should also prove essential in extending the framework. Research on the role of Dynamic Geometry Software will also be considered to discuss the importance of the understanding of function in computer based environment. This focus is important as the facilities provided by DGS such as Geogebra will impact greatly on the activities designed in the study. This review should in turn serve to highlight a gap in the literature where this study may fit in. Additionally, I will focus on review of the various researches focus on the understanding of the concept of function closely related to this study.

- - *The difficulties of understanding function*

Function and in particular linear equation is difficult for students to understand and conceptualize the notion of function and its associated properties. This difficulty and superficial understanding of the function are well documented in the extensive research in mathematics education. (see Elia and Spyrou, ٢٠٠٦; Saika, ٢٠٠٣; Zaslavsky et al., ٢٠٠٢; Knuth, ٢٠٠٠; Markovits et al., ١٩٨٦; Moschkovich et al., ١٩٩٣; Schoenfeld, et al. ١٩٩٣). For instance, Sajka (٢٠٠٣) draws on the precept theory and identifies three kinds of the difficulties of understanding function and a misinterpretation of the symbols used in the functional equation. Although he describes important changes in understanding of the concept of function and symbols. Like similar study, findings of his research indicate students` difficulties in using some symbols occur in algebraic representation of functions. Also Knuth (٢٠٠٠) points out the lake of understanding of the connections between different representations of function in high school and concludes that many students have superficial understanding of the connections between the algebraic and graphical representations even for familiar routine problems.

In the case of line and particularly the slope of a line, Zaslavsky et al. (٢٠٠٢) report confusion of slope, scale and angle concerning the connection between their algebraic and geometric aspects when some common but undeclared default assumptions about this connection are undermined. Their study highlights the importance of the angle between the horizontal axis and the graph of a line and its relation with the concept of slope. Schoenfeld and his colleagues in three studies (١٩٩٠, ١٩٩٣) provide rich explanation on students` understanding linear equation and difficulties students encounter in the process of learning and interpreting line and its graph when the function group at Berkeley developed computer-based microworlds, collectively called GRAPHER, for exploring aspects of mathematical functions and graphs. Interestingly, Pirie and Martin (١٩٩٧) review the literature of different methods and approached for teaching linear equation and present some of the results of a case study of teaching mathematics with meaning to less able pupils. They compare the results with other researches and claim that many problems referred to the literature were not present.

-̃ - *Linear equation as a function*

One of the important domains in mathematics is function and its graphical representation. Functions have been seen at most mathematics curriculum or textbooks, and have been identified as the single most important notion at all levels of schooling. Although function is directly taught in secondary or high school, but the concept of function is used for presenting and teaching other mathematical topics even in primary school. For example, linear equations and its graph are taught as a function and therefore in literature research on this mathematical subject is a part of function domain. There exists a wide range of research considering the teaching and learning of function and in particular linear equations, highlighting a range of approaches and perspectives, illustrating a variety of significant cognitive issues and presenting different obstacles to the learning of linear equations with understanding. For two reasons the topics of function and its graph is of cognitive interest in mathematics education research: ١) function is a multifaceted mathematical concept and ٢) the potential of computer based environments for enhancing learning instructions in this area (Schoenfeld, et al. ١٩٩٣).

- - *Theoretical framework*

An important factor of functions is the diversity of representations and their connection related to the concept of function. A wide range of researches have studied the role of different representations on understanding and interpretation of functions (Gagatsis and Shiakalli, ٢٠٠٤; Hitt, ١٩٩٨; Markovits et al., ١٩٨٦). The concept of function has the capability to admit different supplementary representations, while each of these representations explains particular aspects of the concept without being able to describe it completely. The literature illustrates functions in multiple representations, such as mapping, tables, graphs, and equations to promote and enhance students for deeply understanding the concept of function. Thus, an important educational objective is that students should use different forms of representation of functions efficiently for the same situation and perceive the connection between them.

Another factor is the perspective which the function and particularly linear equations are seen or operated on. As indicated in the introduction, these two perspectives are process and object perspective and offer the potential to update pedagogical decision making in the area of teaching functions (Kieran, ١٩٩٣). Sfard (١٩٩١) historically analyses wide rang mathematical definitions and representations of function through psychological lens and shows that abstract concepts such as functions can be understood operationally (as process) or structurally (as an object). These two ways are fundamentally separated from each other. Sfard (١٩٩١) contrasts the distinction between these perspectives and states:

> . . . Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing- a static structure, exiting somewhere in space and time. It also means being able to recognize the idea “at a glance” and to manipulate it as a whole, without going into details . . . In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed. (p. ٤)

Studying competently on function domain involves at least in two dimensions. One dimension refers to different representations and another one relates to the perspectives.

These two aspects are presented in figure ٢-١ and it can make a framework for understanding and interpreting students` solutions for activities and problems. Students` solutions may reside within one cell of the table or move across representations within one perspective or move across perspective within on representation and even both dimensions (Moschkovich et al., ١٩٩٣). This framework also help researcher to design appropriate activities based on dynamic software to enhance students to construct their own knowledge on linear equation.

	Representation		
Perspective	Tabular	Algebraic	Graphical
Process			
Object			

Figure ٢-١- Theoretical framework for study

The rest of this chapter consists four sections. The first section of the theoretical framework explains the essential role which multiple representations have played in the process of teaching and learning. The second section focuses on review of the various researches which concentrate on the understanding of the concept of function and are closely related to this study. The third section, additionally, discusses the importance of the understanding of function in computer based environment. Finally, in fourth session, reviews important educational issues on understanding mathematics.

- - *Theoretical consideration*

Today most mathematics curriculum and textbooks widely use a range of representations more than ever before in order to promote understanding important and complex mathematical concepts. Using multiple representations has been strongly related to the seeking of students' better understanding in the process of learning mathematics (Elia et al., ٢٠٠٨). The use of more than one representation aids students to achieve a better picture of a mathematical concept (Kaput, ١٩٩٢). For example, Stylianou and Kaput (٢٠٠٢) rightly illustrate that connecting the graphical representation and physical models within the complex dynamic environment can assist students to expand their understanding of intricate phenomena and their underlying concepts.

- - *The Concept of Function in Mathematics Education Research*

The concept of function is fundamentally important in mathematics curriculum and therefore, the mathematics education research have focused on this particular mathematical topics for over the past decades (e.g., Elia and Spyrou, ٢٠٠٦; Gagatsis and Shiakalli, ٢٠٠٤; Sfard, ١٩٩٢, ١٩٩١; Vinner and Dreyfus, ١٩٨٩; Tall and Vinner, ١٩٨١). Several studies have used different approaches to explore the concept of function in mathematics teaching and learning. For example, Sfard (١٩٩٢) introduce the structural and operational understanding of the notion of function. Vinner and his colleagues extensively discuss the concept image and concept definitions concerning students' conceptions of function (Vinner & Dreyfus, ١٩٨٩; Tall & Vinner, ١٩٨١). Blanton and Kaput (٢٠٠٤) suggest that different representational tools (tables, graphs, pictures, words, and symbols) grow with students' age development, to make sense of and express functional relationships; while mathematics instructors traditionally have focused on the use of the algebraic representation of functions. Sfard (١٩٩٢) also shows that students do not make connection between the algebraic and graphical representations of the concept of functions. In most textbooks graphs are traditionally present at the end of a process of learning rather than a mode of representation for analysis, manipulation, and interpretation functions.

By reviewing related researches on different perspectives and approaches of the concept of function such as process and object perspectives presented by Sfard (١٩٩١); Slavit, (١٩٩٧) discusses students` understanding and development of function to present an alternate perspective for employing "the action/process/object framework". This "property-oriented view" of function is based on visual aspects of functional growth and differs from a correspondence view. The study emphasises on the properties of functions which are resulted by changing the variables and provides implications for curriculum developer. Elia and Spyrou (٢٠٠٦) recently draw on both aspects of the understanding of function (concept image and multiple representations of function) show close interrelations among three different components for making sense of functions: first, definition of function, examples in everyday life and second, using different representations of the concept of function. Based on a synthesis of the relevant literature, Elia et al. (٢٠٠٨) in their new study explore and categorise students' display of behaviour in four aspects of the understanding of function: effectiveness in problem solving, defining the concept of function, introducing some examples of function and recognizing functions in graphic form by transferring function from one representation to another one. They conclude that students show better performances in giving examples of function rather than define appropriately the concept of function while problem-solving on functions was observed as the lowest level of achievement. Students successfully employ the concept of function in a variety of representation, to give an appropriate definition and examples of function.

Moschkovich (٢٠٠٤) uses a socio cultural perspective and describes the impact of interaction a student with a tutor to work with linear functions and how that learner appropriated two aspects of crucial mathematical practices for working with functions. Moschkovich examines the role of tutor and interaction whit the learner which this description helped me to be ready for interviewing students in my study. Karsenty (٢٠٠٢) also designed a qualitative study to investigate adults' long-term memory of mathematical concepts learned in school. The mathematical topic was linear functions and participants were asked to draw a graph of a line such as y=٢x. This study indicates the mathematical notion of linear graphing was not retained, because reconstruction of the mathematical idea strongly depends on the length of mathematics courses taken by the students and the quality of learning-teaching process. In terms of my study, I believe graphing applications and software could influence the mechanism of recalling in terms of reconstruction of mathematical concepts, though it needs more evidence and longitudinal research. The ways which Karsenty analysed qualitative data was a useful evidence and assisted me in order to find an appropriate method for analysing data.

Zaslavsky et al. (٢٠٠٢) provide evidence to demonstrate existences confusions concerning the connection between the algebraic and geometric representation of slope and its relationship with scale and angle. These confusions come up because of some common assumptions, regarding the isomorphism between the algebraic and geometric systems. Analysing data reveals 'analytic' and 'visual 'approaches and some combinations of the two main approaches. In sum, reviewing related literature not only presented a framework of learning functions and particularly linear equations but also points out a ‘gap’ for which my research may fit in it.

- - *Computer and function*

Computer software can present dynamic representations of mathematical objects such as line that students had already experienced only as static objects and it can also present multiple representations allowing students to manipulate mathematical objects dynamically (Schoenfeld, et al. ١٩٩٣). By the increasing use of dynamic software for learning about mathematical topics such as functions, students are usually asked to look at very different figures dynamically and are expected to find a common behaviour through them. However, some studies indicate some pitfalls and conceptual obstacles related to the connection between algebraic and geometrical representations of functions that students may encounter during manipulating with graphic software (Demana and Waits, ١٩٨٨; Hillel et al., ١٩٩٢). Difficulties of understanding the concept of slope are addressed in some studies, mostly concerned the misconception of confusing height for slope (Leinhardt, Zaslavsky and Stein, ١٩٩٠; Zaslavsky et al., ٢٠٠٢). For example, Zaslavsky et al. (٢٠٠٢) treat slope as a mathematical entity to make clear distinction between the slope of a function and the slope of a line, while others embed it in a physical or everyday life context (Nemirovsky, ١٩٩٢, ١٩٩٧; Clement, ١٩٨٩). Rasslan and Vinner (١٩٩٥) indicate "whether students realise that the slope is an algebraic invariant of the line and therefore does not depend on the coordinate system in which the line is drawn" (p. ٢٦٤), while Schoenfeld et al. (١٩٩٣) describe one student's attempts to understand the slope of a line and complexity of this concept through working graphical software in Cartesian plne.

There are also a wide range of researches on the role of Dynamic Software as tool affect the process of learning-teaching mathematics. For example, Straesser, (٢٠٠١) analyses the interactions between geometry as a subject, the dynamic geometry software (Cabri) as a computer tool and the students as user and focuses on changes in the teaching and learning. He concludes that activity integrating the use of DGS deeply changes geometry and the process of teaching-learning mathematics. Even some studies focus on details and features of software to demonstrate their capacities. For instance, Falcade, et al. (٢٠٠٧) take the Vygotskian perspective of semiotic mediation and design a teaching experiment to introduce the idea of function with the use of the Trace tool of Dynamic Geometry Software. They assume that the DGS provides a basic representation of variation and functional dependency and it has potential for constructing the meaning of function in the experience of covariation. Also, Pitta-Pantazi and Christou (٢٠٠٩) report the capability of dynamic geometry learning environment in accommodating different cognitive styles and how it may enhance students' learning. Lagrange (٢٠٠٥) studies a functional relationship with help of technological tools based on the use of non-symbolic software like dynamic geometry and spreadsheets and compares them with the use of computer algebra packages.

:- *Understanding mathematics*

It is an accepted idea within the mathematics education community that students should understand mathematics. A considerable number of efforts within the cognitive study have tried to define and illustrate understanding mathematics concepts precisely with multiple representations such as function (see Schoenfeld, et al., ١٩٩٣, Sfard, ١٩٩١; Elia et al., ٢٠٠٨). This study focuses on students` understanding to describe and explain this understanding and attempts to provide a fine–grained level of how the elements of the teaching–learning process interact (Hielbert and Carpenter, ١٩٩٢). The notion of connected representations of knowledge about particular topics such as linear equation and specially the slope of a line provide a useful way to think about understanding mathematics. A mathematics idea or procedure is understood if it is involved in recognizing relationships of similarity and different representations forms of the same concept. A useful way of describing this understanding is interpretations of students` learning to explain changes and reorganization in networks of connection of different representations while new connections are formed, and olds may be modified.

٣- Method and Methodology

٣-١- *Preface*

There are large numbers of researches regarding using dynamic geometry software in teaching mathematics. Most of these researches have focused on geometry and some of them on algebra. In this study, I consider relationship between these two mathematical areas. I believe both Geogebra as a tool and linear equation as a mathematical topic are suitable for this study. In particular, I focus on relation between geometric interpretation of the slope of a line and its y-intercept and some problems represented with linear equations algebraically. I believe teaching the concept of slope by dynamic software is easier than blackboard or paper and pencil, because teachers can present lots of examples in little time and by using dragging mode in dynamic environment, one could make sense of continuous changes in different examples rather than discrete, separated examples. I can use any dynamic geometry software for this study, but I choose to work with Geogebra, because it is translated in Farsi and I have used it in my math classes that I had taught.

٣-٢- *Sample*

For this study, four students from ٨th grade who had learnt linear equation recently were selected and two students from ٩th grade who passed linear equation in previous grade and recently they had studied this subject again in a wider range. This sample represented students with different level of mathematical abilities (low, high and moderate) and different genders (٣ girls and ٣ boys). I chose students who had this topic before because the aim of this research is to investigate in what ways the use of Dynamic Geometry software can change and enhance pupils` understanding in a particular mathematical topic such as the slope of a line and its y-intercept. It should be mentioned that learners had basic knowledge of the computer, and therefore there was no need to teach them computer skills.

٣-٣- *Initial interview*

The students were initially interviewed in order to indicate their basic knowledge and understanding of the line and particularly the slope of a line and its y-intercept based on selected theoretical framework. The interviews were conducted based on three questions posed to recognize pupils` understanding of the slope of a line and its y-intercept. During the interview, students` solutions for each question were discussed with them in order to find their difficulties in understanding the concept of slope. The interviews took place in students` classroom in Iranian school in London. Each interview was recorded by audio tape but this did not interfere. Students were aware of the setting when the interviews started, and they were careful to behave appropriately. Discomfort due to the environment appeared to fade soon because the interviewer (researcher) was their teacher and students were familiar with him. Interviewer gave only very general guidelines for students during the interviews and computer work sessions. Interviews were done to firstly find pupils' misunderstandings in the concept of slope of a line and secondly to find what kinds of activities might be necessary to remedy those misunderstandings.

۳-۳-۱- Questions for interviews

Considering the number of students in sample, I designed a case study research. Initially, I conducted semi structured interview based on three questions about the slope of a line and its y-intercept to identify pupils` understanding and knowledge. Each question is related to one or more aspects of the concept of slope and the levels of knowledge. The first question is at macro organization level (level ۱) and ۲-slot schema. A line has equation y = mx + n if, and only if, its slope is m and its y- intercept is n. The desired line has the same slope as the given line so its slope (m) is ۲; and y-intercept (n) is ۰, because the line goes through the origin, then its equation is y=۲x.

۱- a) On the axes above, draw a line parallel to y= ۲x – ۵ that goes through the origin ۰.

b) On the line below, write an equation of the new line

Equation: ---

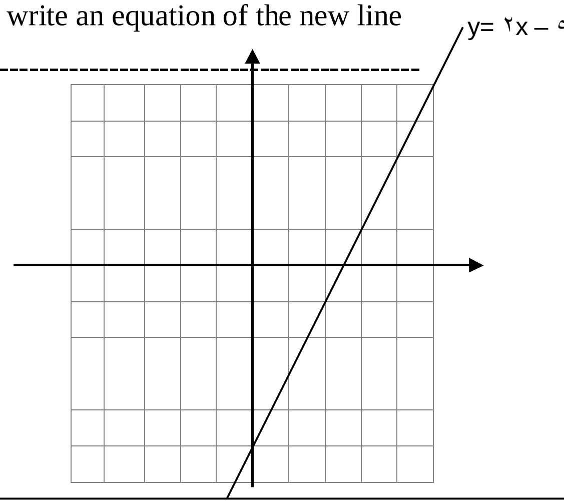


Figure ۳-۱ - Question ۱

The second question is at concept level (the entities that fill the slots at level ١) and their associated properties. In line $y= mx + n$ If $m > ٠$, the line rise, if $m= ٠$, the line is parallel to x-axis, lines with $m < ٠$ slant downward to the right, and lines of large slope are nearly vertical. Also, if $n > ٠$, the line intersects the y-axis above x-axis at point (٠, n) and lines with $n<٠$ intersect y-axis below x-axis. The aim of this question is to find out about students` understanding of the concepts and properties of the slope of a line and its y-intercept.

٢- Which figure does show the line $y=mx+n$ if $m<٠$ and $n>٠$?

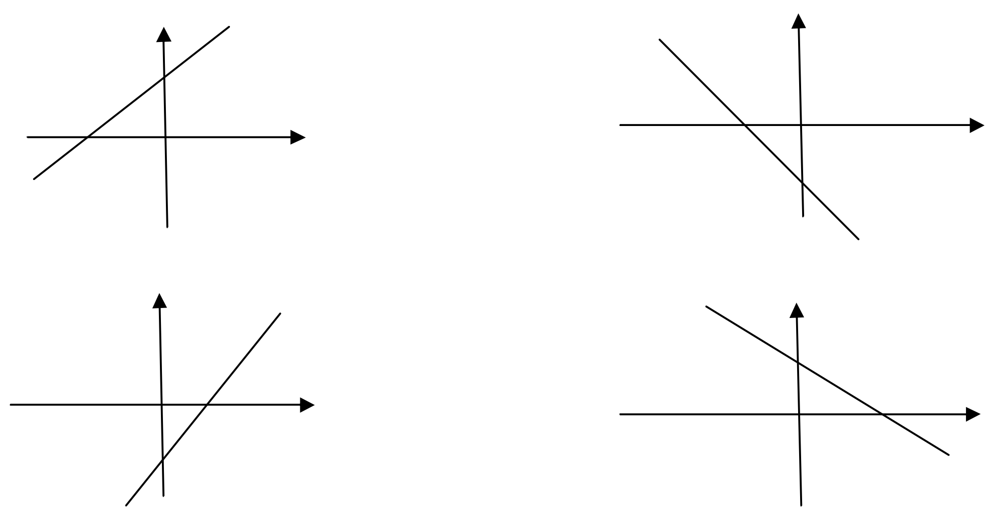


Figure ٣-٢ - Question ٢

The third question is at fine-grained knowledge structure level or Cartesian connection. The aim of this question is to finde the slope of given line by choosing two points on the line and calculating the value of m by using the formula $m= y_{٢}-y_{١}/x_{٢}-x_{١}$. This ratio indicates direction (e.g. positive indicating up and right) and steepness of the line (y increases if x increases and wise versa). In $y= mx +n$ form, if $x=٠$, then $y=m(٠)+n =n$, so the point (٠, n) is on the line, and on the y–axis, hence n is the y – intercept. Also students could have solved this problem by using algebraic method. The line passes through the point (٠, ٣), then the value of n is ٣. The point (٣, -٢) is on the line, hence this point has to satisfy the equation $y=mx+٣$, so $-٢= m (٣) +٣$ and therefore the value of m is $-٥/٣$. In both method students have to select points with integer coordinates.

٣- The figure below shows the graph of the line y=mx+n.

Find the value of m and n.

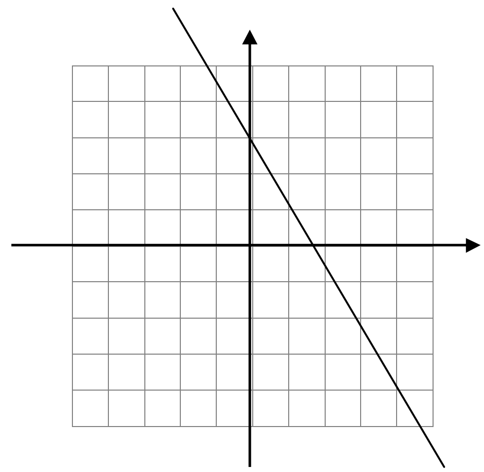


Figure ٣-٣ – Question ٣

˜ ˜- - *Interacting with computer*

In second step, students were asked to use the computer set of tasks within Geogebra software. This was done in two school sessions. The first session held in group, mainly focused on students` familiarity with the software; exploring and introducing the different menu options as well as observing tutorial presentations built into the Geogebra program. Researcher introduced Geogebra as a computer tool for learning mathematics, starting with selecting points and provided a brief technical tutorial on how to use the software. There were no planned activities and no explicit instructional goals in this session. The idea was to let students to experience with this type of software and get interested in it and for the researcher to identify issues for more structured analysis.

In next sessions, researcher worked with pupils individually. The aim was for the pupils to explore the meaning of the concept of slope by interacting with computer based activities. These sessions were observed and notes were taken by researcher. Following the completion of this session, students were asked to do three tasks based on new conceptualization of the concept of slope of a line. These tasks were designed to assess students' understanding the slope of a line and y-intercept and to deepen their understanding. My initial goal was to see how students used the software and what they learnt from activities. Thus, I just participated as a facilitator and interviewer. As facilitator, I made sure the software ran and helped students to set up the tasks they were asked to do. As interviewer, I asked for explanations in order to assess the nature and depth of their underlying understanding of the topic of study. Of course, when students had technical difficulties with the software, my interventions were limited to give little help to assist them for returning to the activities. Only when they appeared to have very serious difficulties, I provided direct tutorial for graphing.

٣-٤-١- Designing activities

Students' cognitive structures and the way they come to understand the concept of slope a line and its y-intercept is important for designing computer based activities. These activities provided the opportunity for researcher to observe not only how well pupils have learnt, but also what they have learnt. Activities should be developed so that they provide a rich learning environment for students to investigate and promote their understanding. So one should have in mind that the computer and software by itself cannot help students without designing appropriate tasks and "tools can be sterile objects unless we provide for appropriate activities with them including designing them and using them in contents that exercise and measure the tool`s power" (A. diSessa, ١٩٩٠, p. ٣٢٢). My concern in designing educational artefacts activities is how students use the technology for learning mathematics. Thus, computer based activities were designed with these five assumptions in mind:

١. The activities should facilitate the process of conceiving complex concepts and their relationships and help students move from the basic level of knowledge about lines in the Cartesian plane to the advanced understandings described in introduction (Schoenfeld, ١٩٩٠).

٢. The dynamic and interactive nature of the software as a medium should be used to make explicit mathematical ideas that must be conceived.

٣. Multiple representations of the function and change in one representation due to manipulating the other should be simultaneously accessible (Schoenfeld, ١٩٩٠).

٤. The computational power of the medium helps both teachers and students to perform computations and they will have time to focus on the conceptual issues being explored which were not available a few years ago and practice exercises faster than on the blackboard or paper.

٥. “The activities as medium should encourage and facilitate students` reflection on their understandings and communication of those understandings” (Schoenfeld, ١٩٩٠, p. ٢٨٨).

More specifically, these activities should enhance students to move fluently across variety representation of the line to understand the concept of slope and y-intercept and should be understandable for ١٤-١٥ year pupil whit a simple presentation, but attractive and should involve students to suggest hypothesis about the slope and its properties. Each activities helps students to observe and learn one or more aspects of the concept of slope and its properties associated based on cognitive knowledge structure. Also, these activities` aim is to remedy and treat students` misconception and misunderstanding about linear equation observed in initial interviews and discussions in their classroom. My experience as a mathematics teacher helped me to design these activities and predict students’ interactions and their needs. In nest section I will try to explain particular rationale for each activity based on my experience in designing computer and dynamic based environments.

٣-٤-٢- Activities and tasks

In individual sessions students used activities. Each activity provided opportunity for students to explore one or more aspects of the linear equation and particularly the concepts of slope and y-intercept.

٣-٤-٢-٢- *Activity* ٢

Initially interacting with computer was started by moving a point on a given line (see figure ٣-٤). Students were able to see coordinates of point C, change the line and simultaneously and observe the change in equation of line. The aim of this activity was to help student to get more familiar with the relationship between two coordinates of a point which was symbolized by an equation. Three representation of a line (tabular, symbolic and graph) are presented in this activity. Although it is expected that all students to be able to find a point on the Cartesian plane knowing its coordinates and also find coordinates of given points, but initial interviews indicated that some students were not able to do this. They also did not have enough knowledge about how to find the relationship between x coordinate and y coordinate in order to write the equation representing this relationship.

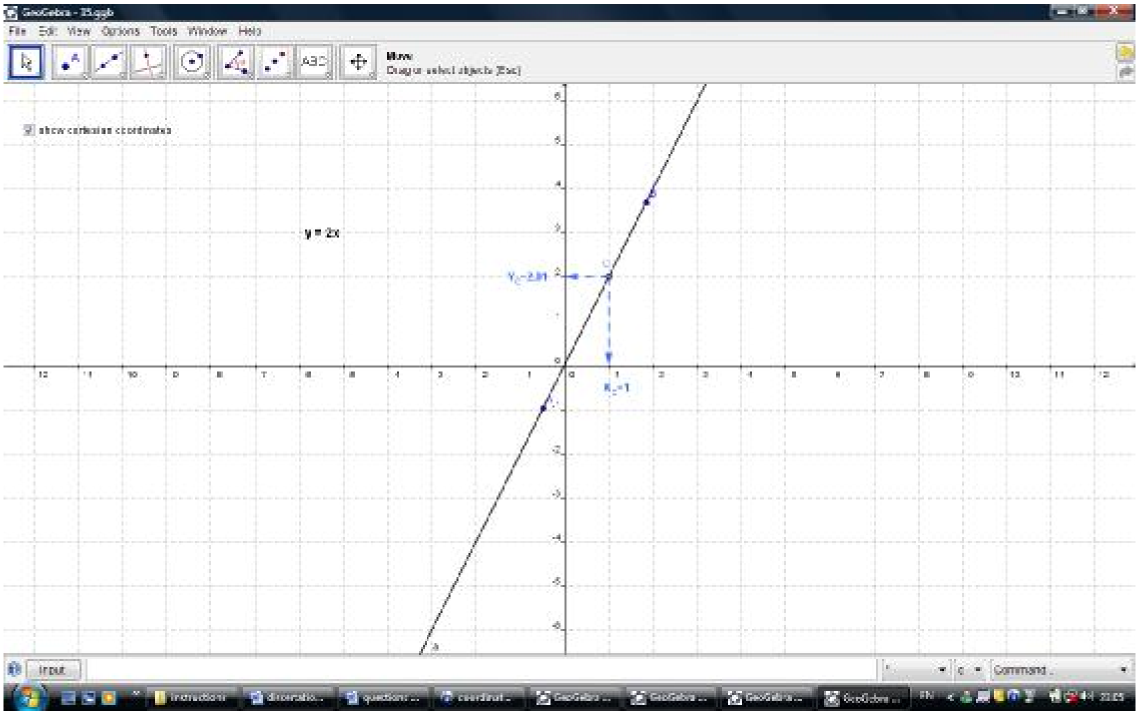


Figure ٣-٤ – Investigating the relation between x and y

- - - -Activity

In next step students were asked to do the second activity (see figure 3-5). They changed the value of slope (a) or y-intercept (b) and simoltaneously observed the chang of the graph of a line on the screen. This activity provided oportunity for them to see the slope and y-intercept as an object in the concept level and investigated their assosiated properties. Students were able to see lines with negetive or positive slope as well as steeper, vertical or horisental lines. They also investigated the relationship between y-intercept and the point where the line intersects y-axis. I designed this activity because students recognized the value of slope and y-intercept in form y=mx+n and wrote the equation of line when the value of m and n were known, but they could not imagine the graph of the line and its position. They also had no image of the graph of line when the value of m or n was negative or posetive. Students had no idea about the magnetude of m, for instance, the line makes smaller angle with x-axis if m is between zero and one. I believed that this activity would help students to make a graphical sense of the value of slope and y-intercept and provide oportunity to explor their associated properties.

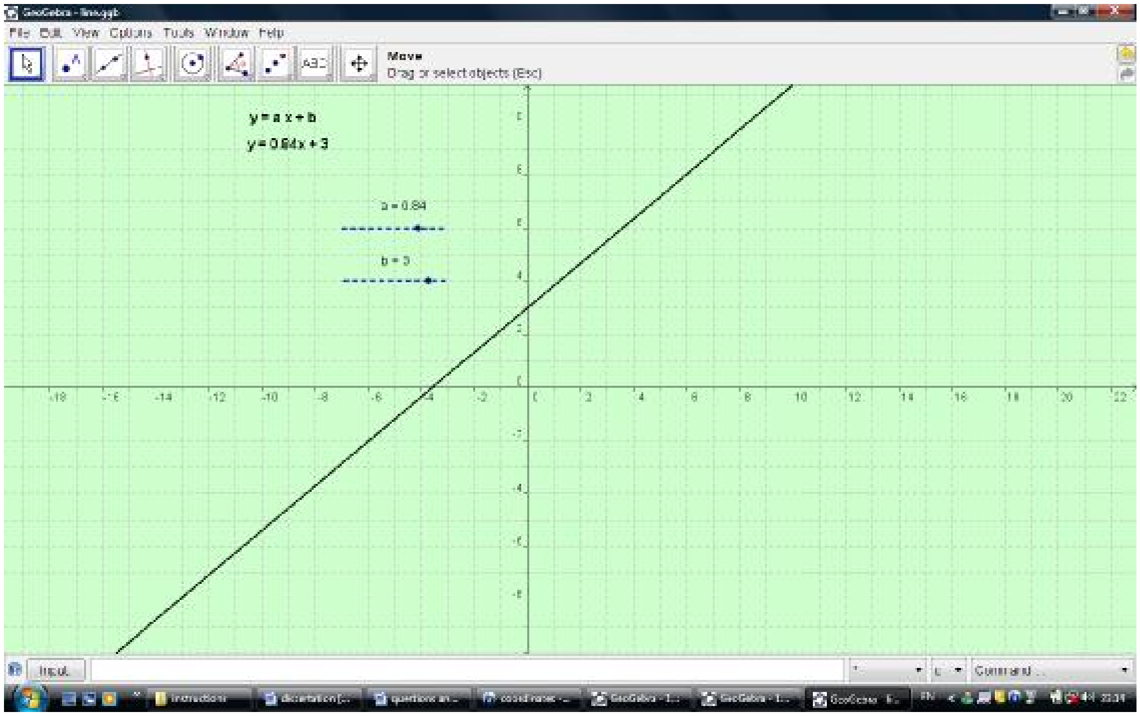


Figure ٣-٥ – Exploring the slope and y-intercept

- - - - -*Activity*

The two parts of this activity helped students to explor the graphical representation of the slope and y-intercept of lines in detail. In first part the activity (see figure 3-6) they chaged the value of y-intercept of three parallel lines where slopes were constant.They found that the lines with equal value of m are paralell and explored the position of lines in the Cartesian plane due to the change of y-intercept. The aim of this activity was also to show the relation between y-intersept and value of "n" in the equation, particularly to pay attention to the position of the line with resopect to negative or positive "n"s. In second part of this activity (see figure 3-7) students manipulated the value of slope for three lines with equal y-intercepts. They found that all lines whith equal y-intercepts but different value of slopes pass through one point on the y-axis and also explored the decreasing /increasing property of lines according to the sign of m. The main aim for these two activities is to provide oportunity for students to investigte independency of the value of slope of a line and its y-intercept.

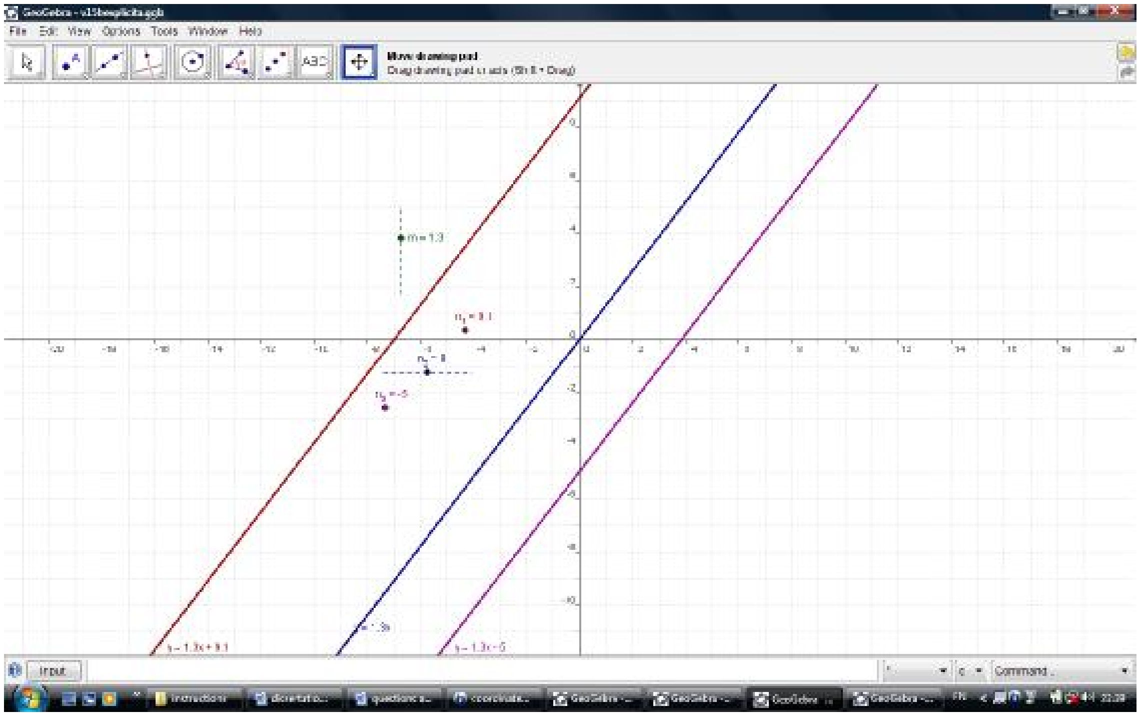


Figure ٣-٦ – Exploring the slope of a line (parallel lines)

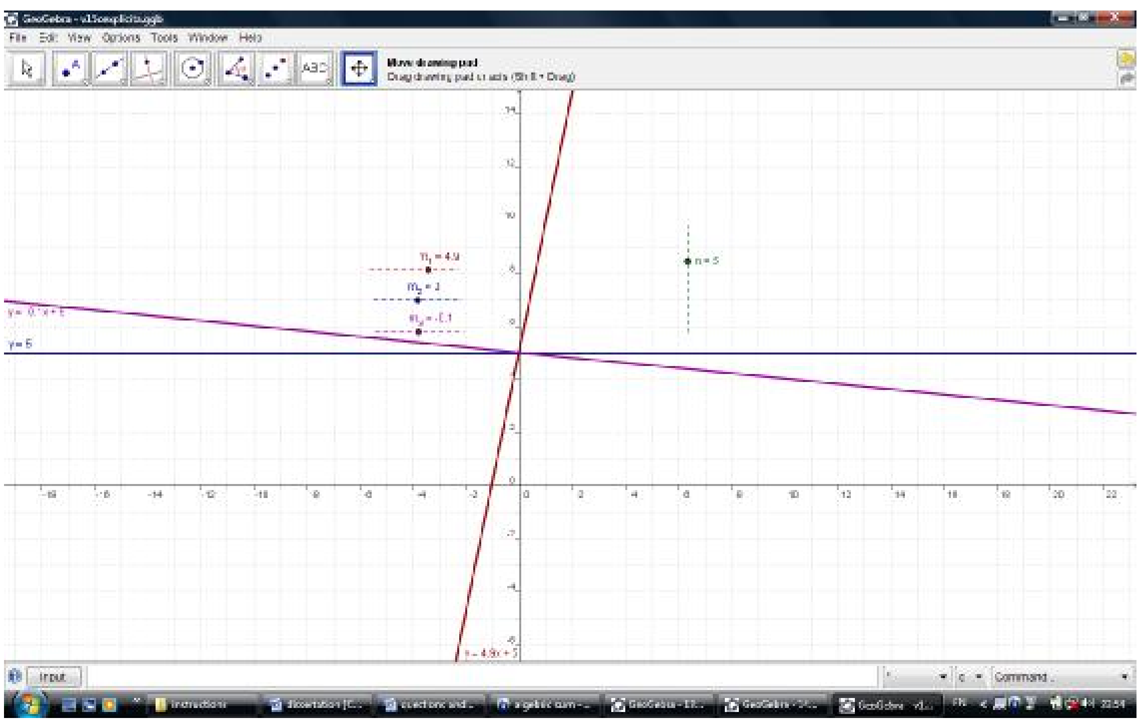


Figure ٣-٧ –Exploring the y-intercept of a line

- - - -*Activity*

In fine-grained knowledge structure level students were invited to work on two parts of this activity. Firstly, they became familiar with the nature and formula for calculating the value of slope in a graphical representation (see figure ٣-٨). They were able to select two points on the line and construct a right angle triangle to show a graphical meaning for the slope of that line. By moving two selected points similar triangles were constructed, but the value of the slope remind constant. Also, the line was changed by dragging and moving its graph. The aim of this activity is to make a graphic and geometric sense of the concept of slope and to provide opportunity for students to manipulate different values in the formula for calculating the value of m in dynamic and graphical representation. Three students who studied in grade eight were not familiar with the formula for calculating the value of m and I expected that this activity could help them to learn and understand the formula and to be able to use it in related problems as well.

Secondly, students were involved in sketching a line with given equation (see figure ٣-٩). Prior to this activity, students were able to sketch a line by selecting two points on the Cartesian plane. This activity was designed to show students that they can draw the graph of the line just by knowing y-intercept and slope of the line. Students were expected first to locate the y-intercept as one of the points and second in order to find the next one, to imagine a right angle triangle where the ratio of the vertical side to the horizontal side is equal to the slope of the given line. For example, figure ٣-٩ shows the line y=٠.٢٥x+١. The first point is (٠, ١) according to the y-intercept and the second point is (٤, ١), because there is a right angle triangle has been constructed with vertical side (١) and horizontal side (٤) and then the ratio is ٠.٢٥. The grid lines on the Cartesian plane help students to count the units and find the value of two sides. The main aim for these two activities is to teach alternative method for graphing a line. They normally sketch a line by selecting two required points on the line and finding their coordinates, but this activity helps students to foster flexible thinking and see the different methods and solutions. Students also learn to write an equation for desired line by finding the value of m and n from graph of the line. I believed these two activities could help students to view the slope of a line and its y-intercept differently.

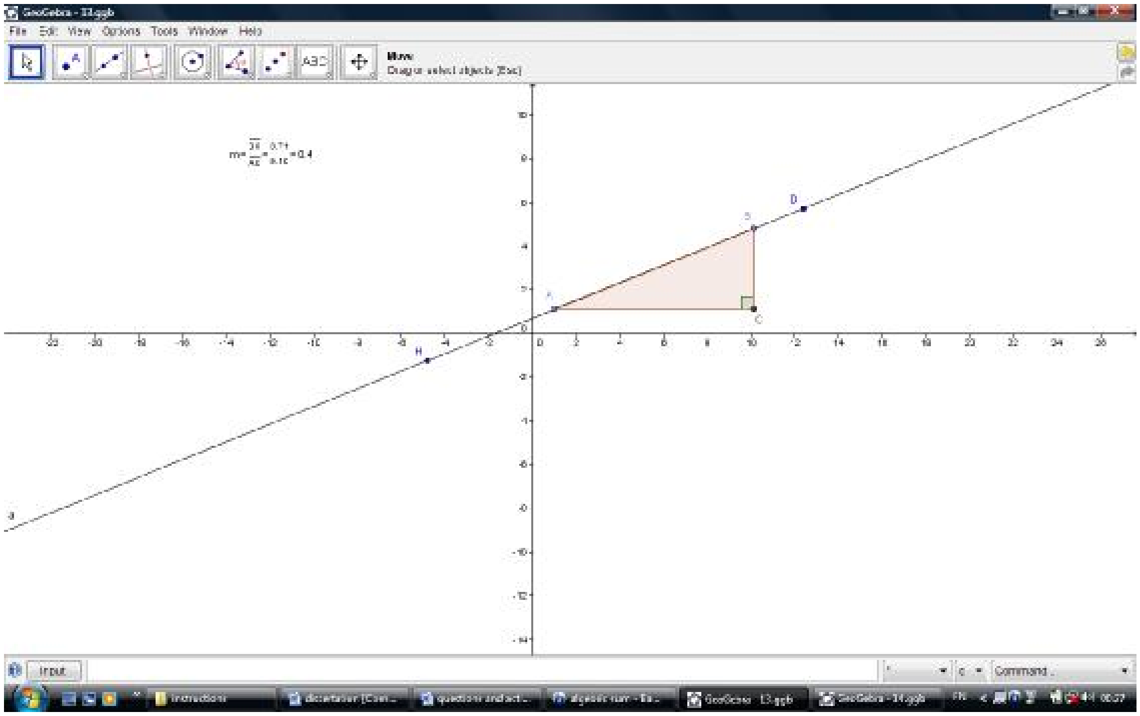


Figure ٣-٨ – Calculating the slope of a line

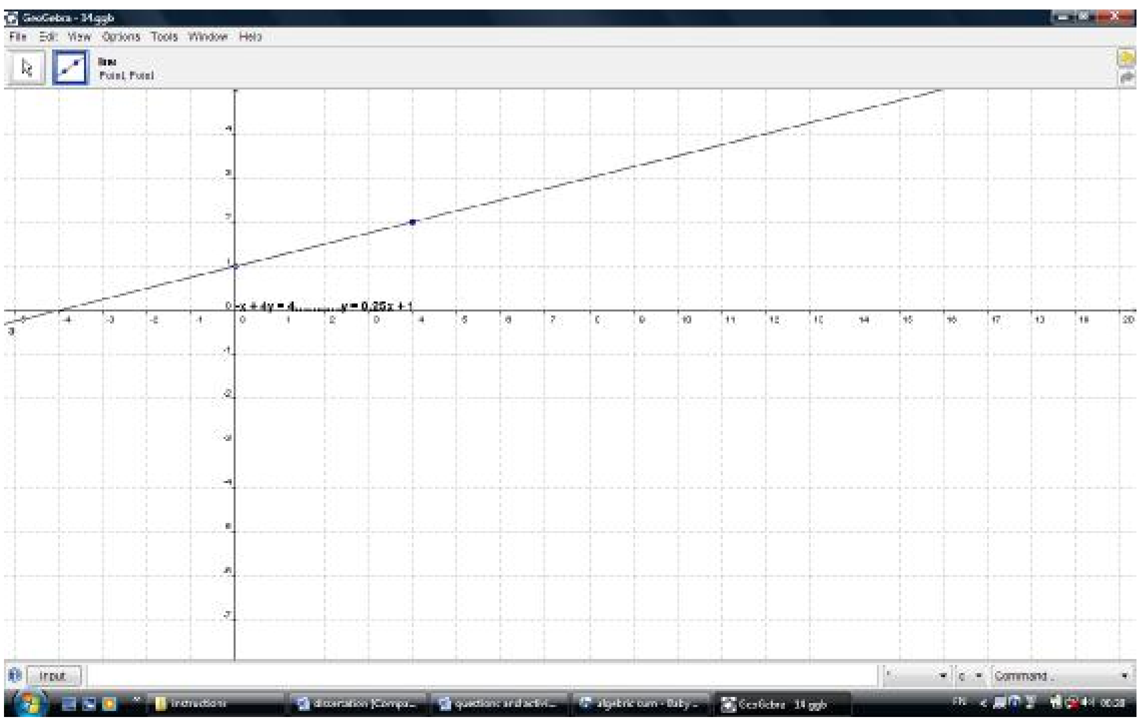


Figure ٣-٩ – Sketching graph of the given line

- Tasks

After carrying on these activities and pupils became confident in graphing the given lines, they were asked to do these three specific tasks. Students had to input the equations of lines in order to draw desired shapes. These tasks aim to provide opportunity for students to apply their abilities and knowledge in order to demonstrate their understanding of the concept of a line and its y-intercept. In several occasions, in order to do the tasks, students felt the need to go back and repeat activities carefully and learn more features of dynamic software such as tracing an object on the screen. Most of students had to redo the activities and they tried to find new insight to help them to draw these shapes. For example, they pay more attention on activity ٢ to find the value of slope of lines which were almost parallel to x-axis. I used these tasks to assess students' learning. Students' accomplishment in doing these tasks completely ensured the researcher that students had learnt the concept of slope and y-intercept.

١- Draw this figure

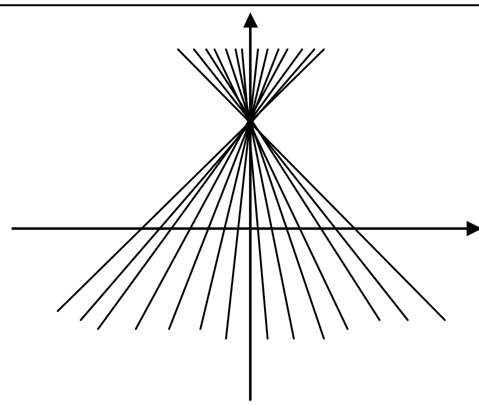


Figure ٣-١٠ – Task ١

٢- Reproduce the Starburst (figure below)

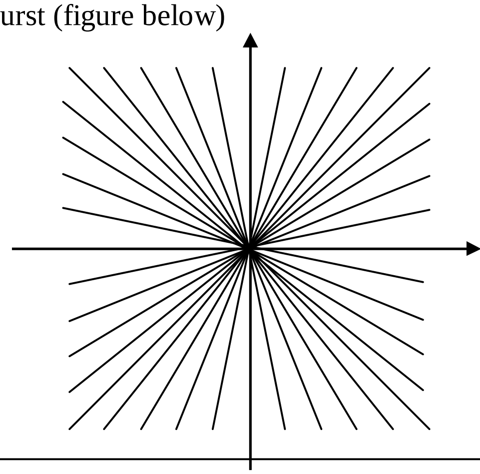


Figure ٣-١١ – Task ٢

٣- Draw the shaded region of the figure below.

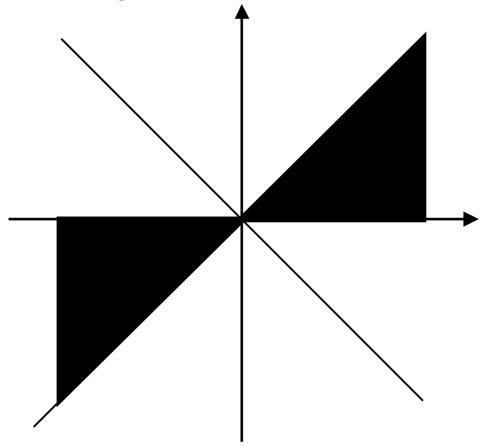


Figure ٣-١٢ – Task ٣

˜ ˜- - *Final interview*

Finally, and in third step, students were interviewed and responded to three initial interview questions again to detect a possible improvement and change of understanding linear equations. If necessary, to gain additional insight and to assure the desired improvement, more similar questions were asked. After careful observation of working sessions, audio-recording and transcribing all interviews and preparing observation schedule a qualitative method were used to analyse the data collected for this study. This empirical study took place over a four-week period in May ٢٠٠٩. This period consisted of one week for first interviews, one week for working with software and engaging with activities (group and individual sessions), and one week for final interviews which took place after one week interruption (at least ٦ days). This interruption seemed necessary to show that students` understanding is lasting, at least in the short period of time.

˜ -˘ - *Data collection*

The data consisted three parts. First, pupils` responded three questions about the slope of a line and its y- intercept. This data were collected after transcribing initial interview. Second, students worked with dynamic software and completed activities and tasks. The data consisted of students' views as well as researcher's observations and notes were taken and also all sessions were audio recorded in order to provide useful information regarding students' conceptualizing mathematical concepts. Third, students were interviewed for the second time and they answered same three questions again. Data collected at this stage could reveal possible change and improvement in understanding a topic in mathematics after interacting with dynamic software and showed in what ways this software could help students to change their understanding of the slope of a line and y-intercept.

The qualitative data were collected by focusing on students' interviews and their responses to three questions before and after interacting with digital technologies in an attempt to make a comparison. Qualitative data also were collected during observing on the learners' interaction with the technology in solving and doing the tasks designed in dynamic environment. Interview and observation schedule were the two instruments of data collection. The interviews which included mostly open-ended items reflected the students` responses and the observation schedule used to document activities of learners. The interviews helped the researcher in the empirical study to identify not only the students' answers but also their process of thinking and possible misunderstanding or misconceptions. Interviews schedule also used to address any unplanned, unexpected actions reported. All interviews were audio-taped to facilitate a more in-depth analysis. Observations schedule help the researcher to grasp as fully as possible the nature of the change in students` understanding of the slope and y-intercept influenced by the computer based activities in dynamic environment.

۳-٦-۱- Variables

Considering the aims of this research, the independent variable is the use of technology and the dependent variable is students' mathematical learning, specifically depth and breadth of their understanding. This study provides a window on learners' mathematical understanding which influences what is taught and how is taught. Is the possible improvement in students` responses only related to the using computer or other factors such as more exercising in mathematics class affect the results? To control other possible factors it is important to conduct the empirical study and the process of data collection in appropriate schedule. So the study was conducted during the time when students had no other classes besides the researcher's sessions and therefore they had no chance to do extra work outside the sessions.

٣-٦- Validity and reliability

The first issue that should be considered is the validity of three questions. Do the answers of these questions show students` understanding of the concept of slope and y-intercept and are they valid? Since the students were involved in the empirical study, it is expected that their responses on three questions in second interviews indicate the change in their understanding after using computer and engaging activities in dynamic environment. The second issue was the reliability of students' responses in final interviews. Would their correct answers remain lasting? In an attempt to ensure that questions reflect students' understanding and are valid and reliable, the questions were selected from related studies, where the validity and reliability was already checked.

٣-٧- Analysing data

The data were analysed using mainly qualitative method and empiricist approach. Interviews were analysed after coding transcribed for seeking common themes which is commonly used in small scale research as well as larger projects (Glaser & Strauss, ١٩٦٧). Themes and codes emerged from the data (I let the facts speak for themselves). The process of analysis involved comparing ideas within the emergent data and related literature and providing coherent arguments for generalizing emergent themes. Interviews were semi-structured and sessions involved note taking during using computer based activities. I tried to be completely open and refusing to be swayed by my own preconceptions towards the data, though this is, in fact, an impossible position to attain, but I tried to develop ideas after finding related research to support what I found during the research. This method is particularly useful for small scale and exploratory researches where related researches have been done before. In analysing the responses to questions, two main perspectives (process and object) on the slope of a line were identified. According to the object perspective, slope is a property of the line and does not depend on its representation and in process perspective slope should be calculated or found using either the equation or points on the line. In next chapter the process of analysing data are presented in detail.

٣-٨- Ethical issues

This study was conducted based on the ethics guidelines published by the British Educational Research Association (٢٠٠٤). The head of Iranian school in London was informed of the purpose and nature of the research through oral communication. School would also receive a brief report at the end of the research process. Students were selected as a volunteer after the aims of the study was explained. They were also offered the opportunity to leave the study at any time. Subsequently a letter sent to their parents to gain their permission for pupils to participate and audio-recorded after informing them about the whole process. All pupils` related documents and audio tapes would be destroyed after submitting dissertation and receiving the grade.

٤- Results and analysis

- - *Preface*

A realistic picture of the understanding that students developed in dynamic environment is presented in this chapter. I developed a methodology (borrowed from Schoenfeld et al., ١٩٩٣) for analysing data, in which the student interpretation at time T is subjected to be checked against their prior responses in order to show a possible change in students` understanding the concept of slope and y-intercept. In other words, an incident that clearly serves as the base for the change of students` understanding is compared with their initial knowledge. To claim about their new knowledge I present detailed, coherent, and reliable analysis of the evolution of students` understanding. A major aim of this study is to explain, as comprehensively as possible, the state of students` understandings throughout their interactions with Geogebra and interviewer. I had to find, if possible, a coherent explanation of what I saw and what lay behind it. My standard in this study is to account for every detail of students` answers and tell a coherent description about them to show that their actions during particular time were due to their knowledge and understanding of the topic. When I claim that there was a change in students` knowledge state at any time, I attempt to explain how and why that change took place. I illustrate the process of change with submitting a small segment of tape.

- - *The case study*

٤-٢-١- Case of HF

HF was the best student in mathematics in grade eight. She is clever but a bit shy and interested in mathematics. She is confident in written exams and mathematical exercises, but she cannot explain her solutions and ideas orally. In first interview, her answers indicated that she has known the concept of y–intercept but she thought that y–intercept and slope of a line are dependent. For example, when she was reading the question ٢, she said m < ٠ means the slope is negative and therefore y–intercept is positive and it is not necessary for this problem to have n > ٠ as condition. She found the value of y–intercept in question ١ and ٣ and the value of slope in question ١, but she had no graphical sense of the nature of slope. For example, she showed the x–intercept as the slope of line in question ٣. In sum, she was very confident in using table and algebraic expression. She solved question ٢ by using two numbers instead of m and n which was interesting for me since I did not expect students solve the question by presenting a numerical example. She wrote an equation as an example (y = -٢x + ١, m = -٢ < ٠, n = ١ > ٠) and sketched the graph of line by forming a table and plotting two points ((٠, ١) and (١, -١)) on the Cartesian plane.

She was involved in computer based activities, and reviewed activities rapidly and when she engaged in drawing line by selecting two points on the grid lines plane, she returned to initial activities which were done quickly and this time reviewed them very carefully. It happened again when she wanted to do tasks. By careful examination of the activity she saw specifically the numbers on the screen and complained about difference between real number and numbers rounded after computing the slope of lines by the software. At the end of computer work session, she drew voluntarily a complete shape by inputting variety of lines and segments with different slops. Then she found the slopes of lines passing through any two selected points.

٤-٢-٢- Case of BA

BA was in grade eight. He was a typical average student in mathematics. He speaks confidently and answers the questions very fast. He likes computers and has adequate knowledge of computers and works with them properly. He answered question ١ by writing algebraic expression for desired line and sketched it by forming a table and plotting two points on the grid lines plane. He solved question ٣ using algebraic method as well, though he was unsuccessful. He recognized the y–intercept correctly in all questions, but he identified x–intercept as the slope of line. BA`s solution for question ٣ was interesting, though he found wrong answer because he had lots of operational and computational mistakes. He wrote y = mx + n and substituted n by ٣, by counting ٣ squares on y axis (the line intersects y axis on point (٠, ٣)). Then BA selected one point (٣, -٢) on the line and substituted its coordinates for x and y in the equation.

He reviewed all activities very fast but was interested in the last activity and wanted to work more with it. He had to find two points on the grid lines plane to draw a desired line according to the value of slope and y–intercept, but he tried to find two points in his mind by selecting a value for x and computing y and without using slope. He understood the nature of slope and y–intercept and learnt them as an object. When I asked him to follow the aim of activity, firstly he had difficulty to find appropriate right angle triangle for computing the slope but after doing enough exercises, he became confident and answered all questions in final interview correctly. He, also drew a complex shape voluntarily by choosing different linear equations and then computed the slope of given lines.

٤-٢-٣- Case of EM

EM was in grade eight and his ability in mathematics was low. He looks quite naughty and careless. His grades were lower than mean in formal mathematics exams. He was not able to answer problems on linear equation presented in his mathematics textbook. E.M. answered all questions in initial interview in a wrong way. He had not enough knowledge about y–intercept, slope and even coordinates of a point. I posed some new questions to recognize his abilities and knowledge about this particular topic in mathematics. When I saw his answers I was disappointed. For example, I asked him to write an equation for a line parallel to the line y = ٢x – ٥ and he wrote: y = ١x – ٥. Then I asked him to write an equation for a line passing through the origin and he wrote: y = ٠ + ٥. He could just draw a table and find two points on the line y = ٢x + ٣ to sketch its graph, even in this case he had lots of operational mistakes.

The process of re-teaching linear equation through computer based activities for E.M. was slow but efficient and his progress was unbelievable. Most of our time was spent on drawing line by selecting two points on the grid line plane according for given value of m and n. This activity helped him to understand the concept of slope and y–intercept, but he was not still sure about his answers. He answered a question correctly, but later he was not able to answer the same question again. Since he could not connect the slope and y-intercept as mathematical entities, he could not understand their meaning and so easily recalled their properties wrongly. Computer based activities have two features to enhance students` understanding: (a) in short time it is possible to do considerable number of exercises; (b) there is a chance for students to correct their mistakes and they receive rapid feedback automatically after each response. These two features, helped researcher to remedy EM`s misunderstandings and misconceptions in one and half hour.

Finally, he made a correct and quite sustained understanding about the concept of slope and y–intercept, when he answered three questions completely after eight days of computer work session. Firstly, he tried to solve question ١ by making a table, but he changed his solution and wrote equation y = ٢x and selected two points on Cartesian plane: the origin (٠, ٠) and (٢, ٤). He found correct answer for question ٢ and also answered two similar questions posed in final interview. He wrote n = ٣ and computed the value of slope (m = - ٥/٣) by drawing an appropriate right angle triangle.

٤-٢-٤- Case of BE

BE was studying in grade nine and her ability and knowledge in mathematics was adequate. She looks clever but careless and feels uncomfortable when involved in mathematics problem solving. She answered all questions in first interview correctly and completely, because she had studied this topic twice: in grade eight and again in grade nine in a wider range. In computer work session she reviewed all activities and tasks very quickly. The final interview and her solutions for three questions correctly indicated this following change interpreted based on the theoretical framework for this study. Dynamic software based environment helped BE to move from process perspective to object perspective. BE`s solutions for tasks and problems moved across representations within process perspective before involving in computer work. In the process perspective BE`s attention is directed to the relationship between the x and y values represented in tabular form (a list of (x, y) pairs) or as an algebraic equation. As Moschkovich et al. (١٩٩٣) noted, students typically do so entirely within the process perspective although it certainly is possible to think about these problems from the object perspective, but Moschkovich et al. (١٩٩٣) suggested that students do not this.

BE`s understanding of the slope of a line and its y–intercept moved from process perspective to object perspective after interacting with computer based activities and tasks. Her solutions for question ١ and ٣ in initial and final interview rightly indicate this change. Firstly, she answered question ١, by drawing a table and selecting two points ((١, ٢) and (٢, ٤)) and writing an algebraic equation y = ٢x, then she sketched the line in the Cartesian plane. She, also, solved question ٣ by using algebraic formula for computing the value of slope. But BE in final interview solved these questions in other ways. Her reasoning for question ١ was that: (a) the desired line pass through the origin then the y–intercept is ٠ and the origin is one point of the line; (b) the desired line is parallel to the line y = ٢x – ٥, then they have the same slope. She selected point (١, ٢) on the grid lines and showed a right angle triangle to prove that the slope of the desired line is ٢. She solved question ٣ by drawing a triangle and counting the squares on the grid lines and found the value of m by dividing the vertical segment on the horizontal segment while her solutions in the first interview involved either finding two points or algebraic methods.

٤-٢-٥- Case of AF

AF was in grade ٩ and his mathematics ability and knowledge was low. She is very shy, and she has difficulty to explain her answers and solutions orally. She does not feel confident in mathematics and has anxiety when she encounters mathematics problems. In first interview her answers were disappointing, because she just answered first question with some mistakes, she needed researcher’s guidance and she had no idea about y-intercept at all. For example, her answer for question ٢ was unexpected. Figure ٤-١ shows her explanation for choosing this answer.

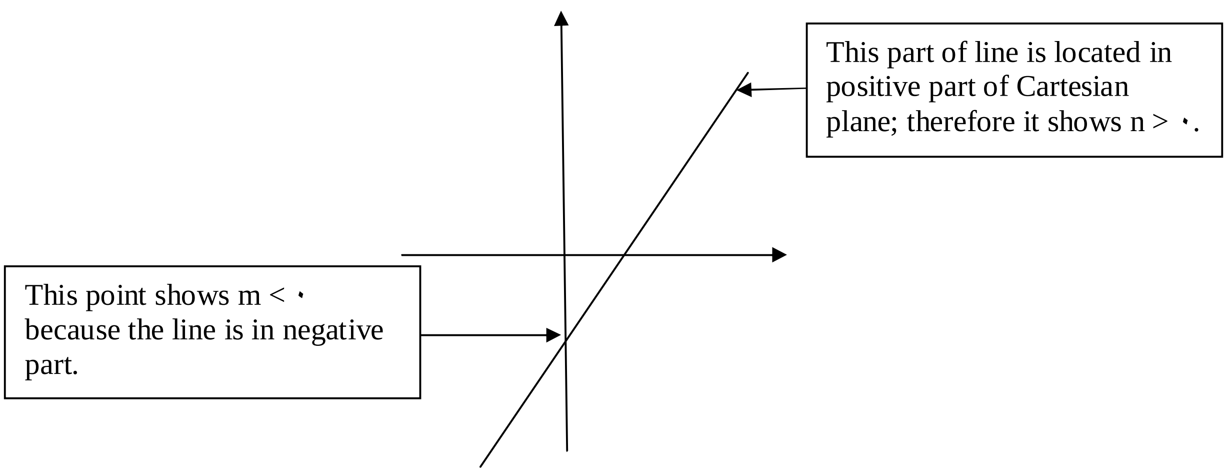


Figure ٤-١ – AF`s explanation for question ٢

The process of manipulating activities and her involvement in tasks was slow, but eventually she made satisfactory progress. She reviewed all concepts about this particular topic of mathematics (linear equation) through engaging with activities and finally she became able to answer three questions completely. She was very happy, felt confident and believed that these activities provided enough opportunity for her to learn this particular topic. She followed all activities carefully and found some mistakes when computer rounded the value of m. For example, she showed that a point has coordinates x = ٢ and y = ٣.٩٩ while the equation is y = ٢x. She returned to the activities when she wanted to do the tasks. For instance, AF started to input linear equations to make Starburst in task ٢. She returned to activity ٢ to find the value of m when slope of lines is between ٠ and ١. Going back and forth with activities and tasks helped AF to construct her knowledge without any direct guidance and tutorial.

٣-٣ - *The levels of Analysis*

An overall view of the levels of analysis and the cognitive knowledge domain that I covered and analysed in this study is presented in Figure ٤-٢ based on Schoenfeld et al.`s (١٩٩٣) study. The most left column classifies three levels of knowledge found in the data. The second column illustrates the standard analysis of the linear equation y = mx+n in the Cartesian plane. These two columns were applied to illustrate students` knowledge structure for the same subject matter before and after they interact with computer based activities. My task in this study was to describe the changes in students` knowledge and understanding and complete two next columns as they interacted with the dynamic environment. The general structure of this table is well-matched to the cognitive psychology literature (see Schoenfeld et al., ١٩٩٣).

Levels	Traditional view of subject matter	Students` knowledge structure before using computer	Students` knowledge structure after using computer
١. Macro organization (schema level).	٢-slot schema: L:y=mx+n Slope y-intercept		
٢. Concepts (the entities that fill the slots at level ١) and their associated properties.	m is the slope of L m > ٠:L rises large ١ m ١ :L sleep (and more) The point (٠,n) is called the y-intercept of L. (etc.)		
٣. Fine-grained knowledge structure: primitive elements and connections among them.	M= y٢-y١ / x٢-x١ are directed line segments, so their ratio indicates direction (e.g. positive indicating “up, tight”) and steepness (so much y for so much x). When x=٠ in y= mx +n, y=m٠+n =n, so the point (٠,n) is a on the line, and (n) on the y – axis. Hence it is the y – intercept. We call this level of structure the Cartesian connection.		

Figure ٤-٢ - Graphs of straight lines: Levels of analysis and structure

According to Schoenfeld et al our knowledge, perceptions and expectation of what we will encounter in the world are organized at the macro or schema level. The most familiar description for straight lines in this level is that a line has equation y = mx + n if, and only if, its slope is m and its y- intercept is n. The second level explains objects and their properties in the related domain. The slope and y-intercept in this level are known as objects not only fill the slots in the schema, but also their properties are readily accessible. For example, If m > ٠, the line rise and if m= ٠, the line is parallel to x-axis. Lines with negative slope slant downward to the right, and lines with large slope are nearly vertical. When n is greater than ٠, the line intercepts y-axis above x-axis at point (٠, n). At the third level, a rich knowledge base tightly connected and serves as the link between the manipulations in the algebraic representation, in which m is simply calculated by the formula (y٢ – y١) / (x٢ – x١), and the graphical representation, in which m has graphical entailments. My study, then, concerns changes in students` understanding as they learned about linear equation and its graph.

~ ~- - *Analysing data*

٤-٤-١- Level ١ – Macro organization

This section describes students` initial schema–based knowledge of the algebraic and graphical properties of straight lines and the evolution of that knowledge as they interacted with activities and tasks. The schema knowledge at this level is known as the slope of a line and its y–intercept to obtain a complete description, both graphically and algebraically. The line L has equation y = mx + n, if and only if, the slope is m and y – intercept is n. This is an algebraic description, but graphically, the line passes through the point (٠, n) and any point on the line moves m units vertically for every unit in horizontal direction to be placed on the line.

Initially, students had non-standard knowledge at this level. They believed that slope and y–intercept were necessary to determine the graph or equation of a line, but four students had not noticed that the y–intercept of desired line (question ١) was zero, because the line goes through the origin. One student was not able to recognize the slope of parallel line, and had no idea about the value of y–intercept and was completely unaware of the ٢–slot schema. Another student was not aware of independency between slope and y–intercept. This student tried to make a connection between these two values to find an appropriate point on the Cartesian plane to sketch the desired line.

Two activities helped students to remedy their misconceptions during the sessions (see figure ٣–٦ and ٣–٧). Students changed the value of the slope or y–intercept, observed the change in graphs of lines and compared the algebraic equations on the screen. They not only saw the effect of changing the value of slope and y–intercept on the graph of lines, but also understood that these two parameters are independent. They found that parallel lines have the same slope and if the value of y-intercept is shown by n then the line pass through the point (٠, n). All students answered the question ١ in final interview correctly and were able to describe their reason for finding the value of m and n and for writing equation and drawing the desired line. It also appeared that students learned function of the slope and y–intercept, so that they used them appropriately to describe their reason for selecting correct answer in question ٢ and to generate the algebraic equations of the lines that they wanted to sketch in question ٣. Both slope and y–intercept were consistently changed in the process of interacting with computer based activities.

٤-٤-٢- Level ٢ – Concepts and their properties

At first glance, the concept of slope and its entailments looks rather simple. The following sentences summarize what one need to know at level ٢ (concepts and their properties) from the traditional view of subject matter. If the equation of line L is given in the form y = mx + n, the value of slope (m) can be read as the coefficient of x and the magnitude of m indicates how steeply the line inclines. If m is near zero the line is close to horizontal and increasing steep for larger values of m. The sign of the slope indicates whether the line goes up or down. The line moves up if it has positive slope and down if the slope is negative. In the form y = mx + n, n is the y–intercept and the point (٠, n) is called the y–intercept of L and the line intersects y–axis at this point.

Observing how much of that knowledge was acquired by students, helps to explain why their performance was unsustainable and why while they recognized m and n as the slope and y–intercept in y = mx + n, it was difficult for them to make sense of graphical representation of the slope and y–intercept. This difficulty was detected in first interviews particularly in students` responses to question ٢. Just one person answered this question correctly. One of the misunderstandings related to these above mentioned concepts was the relationship between the slope of a line and its location in Cartesian plane. As it is seen in the literature, some students believed “that lines of negative slope have the negative direction, that is, they come from the bottom left of the plane and move upwards to the right" (Shoenfeld et al., ١٩٩٣, p. ٨٩). Students thought that increasing the slope of a line would cause the line to move off the screen and they did not consider y-intercept. Thus, another difficulty was that slope sign, slope magnitude, and y–intercept seemed dependent to students and were all related in some unclear way.

As they worked with software and interacted with activities and tasks, many of students` misconceptions surfaced and were corrected at this level. Their confusion about slope sign remedied completely. Manipulating with activity ٢ (see figure ٣–٥) helped them to understand that the slope and y–intercept are independent and make a graphical sense of the slope sign and slope magnitude. Also, three tasks helped students to apply their new knowledge about the slope of a line and its y–intercept. These tasks motivated them to rethink about graphical representation of m and n and most of them return to the activities to see them again but this time with more care. They paid more attention to the change of the graph when the value of the m or n was changing. In task ٣, all students commented about the region where m could change and it became apparent to them that m should be between ٠ and ١.

٤-٤-٣- Level ٣ – Fine–grained knowledge structure

In this level, the slope (m) of line L in the Cartesian plane is a measure of the direction of line and the value of m can be obtained by taking any two points on L, (x١, y١) and (x٢, y٢) and computing m by dividing y٢ – y١ on x٢ – x١. Sign of these two values are indicates direction (e. q. Positive indicating up – right) and steepness (so much y for so much x). When the equation of L is represented in the form y = mx + n, the value of n indicates the y-intercept and when x = ٠ in y = mx + n, y = ٠x + n = n, so the point (٠, n) is on the line. Since the point (٠, n) is on the y–axis as well, it is the y–intercept. This level is called the Cartesian connection (Schoenfeld et a.l, ١٩٩٣).

The third question for semi-structured interview was designed to investigate students` knowledge at this level. Three students who were studying in grade eight were not familiar with the formula for computing the value of m. Although two other students knew this formula, but just one student was able to find the value of m correctly. She remembered the formula for calculating the slope as m = (y٢ – y١) / (x٢ – x١) and knew that the values of m, Xs and Ys in the formula for slope are obtained from the coordinates of two points on the line. She selected two points (٠, ٣) and (٣, -٢) and in spite of her weak in algebra, she was able to use the formula correctly to determine the value of m, which she needed to write an equation for desired lines. One student in grade eight solved the third question in algebraic way. He, firstly wrote a parametric equation y = mx + n. Then he found the value of n and said the line intersects y–axis on point (٠, ٣), so the value of n is ٣. He selected a point on the line (٣, -٢) and said: this point has to satisfy the equation, because this is a point on the line, hence if I put ٣ and – ٢ for x and y, I can find the value of m.

Other students had no idea for solving this question and one of them used the x–intercept instead of the m. All students neither had nor expected a firm connection between the algebraic and graphical representation of functions. They knew that m could be calculated algebraically and could be used in order to write the equation of the line, but they had little idea about the properties of m and n as components of y = mx + n in the plane. The similar researches report the same results (e. q. See Schoenfeld et al., ١٩٩٣). In fact for students the properties of m were linked to their division of the plane into positive and negative areas which caused students difficulties or interfered with their understanding of y–intercept.

The data regarding students` slow change in understanding of slope point to the complexity of understanding elements of the knowledge in it. Focus of this study was on two main themes. Firstly, learning (even simple idea) in a complex domain means making many connections to other pieces of knowledge. The second is the nature of conceptual change particularly in making graphical sense of the concept of slope and y–intercept. Students learned to find the value of m and n from the graph of line and also learned to sketch graph of a line knowing the value of m and n. This change was occurred after their interaction with activities (see figure ٣-٨ and ٣-٩). These activities helped students to find the value of m by choosing two points on the line, making a right angle triangle and dividing vertical side of right angle by horizontal side. These activities also helped them to sketch graph of a line by selecting two points on the Cartesian plane. Knowing n, they selected point (٠, n) as y–intercept and the other point was found by making a right angle triangle according to the value of m. Using this method in order to cross process perspective and move into object perspective was the most important change of students` understanding of the slope of a line and its y–intercept in this level.

The data indicated that this change was prolonged and all students were able to answer third question in final interview.

٥- Discussion and conclusion

˜ - - *Discussion*

The main focus in this study is to delineate the changes in students` knowledge and understanding the concept of slope of a line and its y-intercept. The data confirm aspects of process of this change that I think are important and can be generalized. These themes have been emerged after analysing data over the time students spent working with computer based activities and tasks. There was no curriculum to be covered and activities were designed as a digital medium to support students` learning about lines and graphs. I describe aspects of activities and tasks, those which worked as I expected and those that did not. Also I reflect on general issues of instructional design in technological environments, as well as explaining the key issues of the change in students` understanding after interacting with these activities.

All students were taught that a line has equation $y = mx + n$, if and only if, m is slope and n is y–intercept. They were able to sketch graph of a line by finding two points on the line and making a table. In other words they moved fluently through three cells (algebraic, tabular and graph) in process perspective. This ability was seen in initial interviews, but their understanding of the slope and y–intercept as object were problematic. They preferred algebraic method and used table to answer three questions and they had no ideas of graphic and geometric interpretation of slope and y-intercept.
Designing appropriate computer based activities helped students to understand the slope and y–intercept as an object and move through cells in object perspective. These changes were seen in students' responses for three questions before and after interacting with computer. I try to illustrate these changes in the following sections.

٥-١-١- Y–intercept

The n in the equation form $y = mx + n$ typically refers to the y–intercept but students did not know that the point (o, n) lies on the graph of the equation or the students did not realise that the parameters m and n in this form are independent and one can change without changing the other one, and this generates a family of lines with specific properties. Dynamic software based activities provided opportunity for students to see this property clearly and opened a new window for them to understand the nature of y–intercept. In tasks ١ and ٢, students made a set of lines with different slope and same y intercept. Also, in final interviews their answers indicated that they could find the value of y-intercept (n) in questions ۱ and ۳ and interpreted the meaning of $n < o$ correctly. This change remained prolonged and all students were very confident about the nature of n.

۵-۱-۲- Sign of m

Although, students initially were completely aware of the fact that m in $y = mx + n$ indicates the slope of line but they were not able to interpret the meaning of the slope graphically and show the position of given line in Cartesian plane. In initial interviews they showed x–intercept instead of the slope. For example, they said this line in figure ۵-۱ has negative slope because x- intercept is negative.

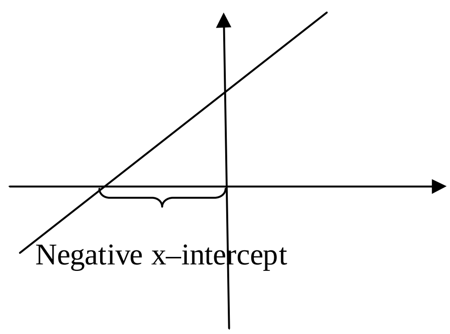


Figure ۵-۱ – Negative x–intercept

By dragging line around y-intercept and observing the change in the sign of m, students became familiar with the negative and positive slope but they were not enough confident when two parameters, m and n, changed simultaneously and they needed more exercises. Dynamic software based activities influenced students` new knowledge about sign of m and made a graphical sense of the slope of a line.

۵-۱-۳- Magnitude of m

All lines of the form $y = mx$ pass through the origin; Comparing L_۱= m_۱*x with L_۲=m_۲*x shows that if m_۲ > m_۱, then L۲ rises more steeply. Firstly, the value of m had no graphical meaning for students. Some of them employed the process perspective by using points on the graphs and determining their coordinates using the equations of the lines. After manipulating computer based activities, students understood the slope of a line from the object perspective and used the individual lines as members of the parametric family [$y = mx$; $m \in R$]. They made a graphical picture of the value of m and showed it correctly on Cartesian plane. Data point to following issue which was raised from task ٣ and when students were involved in shading region of plane between two lines $y = ٠$ and $y = x$. This region is covered by a set of line of the parametric family [$y = mx$, $٠ \le m \le ١$]. It was interesting for them that the function which represents change in slope is not linear, because the other set of lines of the parametric family [$y = mx$, $m > ١$] covers the same area.

٥-١-٤- Nature of m

Given any two points on the line, $(x_{١}, y_{١})$ and $(x_{٢}, y_{٢})$ which satisfy the equation $y = mx + n$, then the magnitude of m is calculated by the ratio of the difference between y coordinates of two points on the line over the difference between x coordinates of two points on the line [$m = (y_{٢} – y_{١}) / (x_{٢} – x_{١})$]. If the ratio is large, then $(y_{٢} – y_{١})$ is large compared to $(x_{٢} – x_{١})$; hence, the graph rises or falls steeply. Similar argument can be used when $(y_{٢} – y_{١})$ is small and $(x_{٢} – x_{١})$ is large. Two students among participants in this study who were studying in grade nine were familiar with this formula, but their performance in initial interviews showed that they did not understand this concept and formula completely, because they were not able to answer the third question and even they did not select any points on the line to compute the value of m. Other students who were studying in grade eight were not familiar with this formula. Therefore, as noted in chapter ٣, an activity was designed to treat this problem and give the opportunity to the others to improve their understanding of what they had taught before. (See figure ٣-٨). This activity was very useful and influenced students` understanding and helped them to make a sense of the nature of slope. The variety of shapes of triangle in dynamic environment made a graphical meaning for the slope. Two form of right angle triangle are presented in figure ٥-٢, to demonstrate two lines with different slopes. Line $L_{١}$ has positive and small value of m and line $L_{٢}$ has negative and large magnitude for slope.

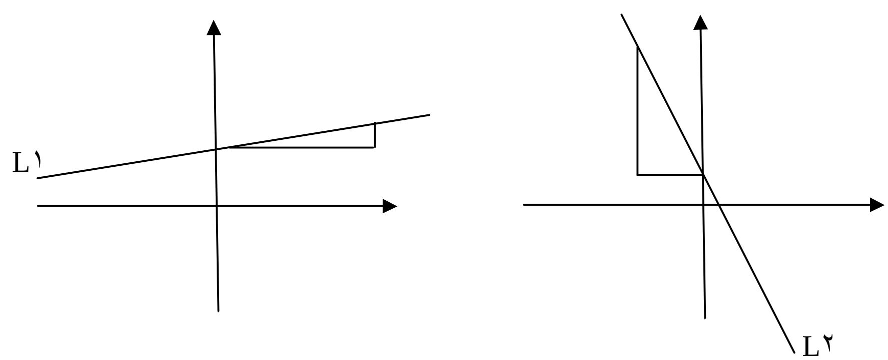


Figure ٥-٢: Two lines with different slope

Forming triangle and computing m by dividing vertical segment on horizontal segment, made a new sense for the nature of slope of a line, especially when students use grid lines for finding the magnitude of two segments. Students had different pairs of points to draw a set of mental similar right angle triangles to compute the value of slope and these similar triangles indicate that the value of slope is constant for any two points selected. Figure ٥-٣ shows a set of two similar triangle selected on line y= ٢x and one can count squares to find the magnitude of vertical and horizontal segments to compute the value of slope (m= ٢) in each triangle.

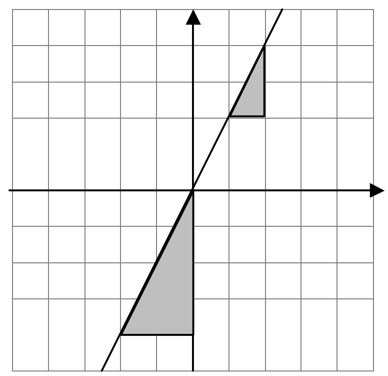


Figure ٥-٣ – Similar triangles formed on line y = ٢x

By imaging these triangles in mind, students not only were able to find the value of m, but also they could make graphical sense of the concept of slope and get a feel for their physical experience such as the slope of a hill or road. Activity ٤ (see figure ٣-٨) helped students to move from process perspective to object perspective and to make a graphical sense of the slope as an entity. Students also understood that the slope of a line and y–intercept are independent by dragging line and moving along the y-axis to make a set of parallel lines. All these parallel lines had the same slope but different y intercept, since they could see that congruent triangles were formed by sliding the original one.

٥-١-٥- Sketching graph of a line

Typically, students` attention is directed to the relationship between the x and y values of a given equation to make a table of (x, y) pairs (at least two pairs) in order to sketch graph of a line on the Cartesian plane. In the process perspective the focus is on the x and y values and the given relationship between them to find the sets of individual points in the Cartesian plane that constitute lines. This method for sketching graph of a line normally is found in formal curriculum and textbooks, but the problem is, students who sketch the graph of line correctly gain no graphical sense of the slope or y–intercept as an entity or object. The last activity (see figure ٣-٩) was designed in order to introduce an alternative method for sketching graph of a line based on object perspective helps students to make a new and geometric sense.

In this method students were invited to sketch graph of a line based on the value of slope (m) and y–intercept (n). It is clear that for sketching graph of a straight line, selection of two points on the Cartesian plane is enough. According to the value of n, the first point (٠, n) is selected on the plane. Considering the value of m second point is found to make a right angle triangle where the ratio of magnitude of vertical side over magnitude of horizontal side of this triangle is equal to the value of m. For example, in figure ٥–٤ graph of the line y = ٣x + ١ is sketched by selecting two points A and B. The point A is defined by n and a right angle triangle shows the value of m by dividing the magnitude of vertical segment (BC = ٣) by magnitude of horizontal segment (AC = ١). The grid lines on the Cartesian plane help students to count the unit squares and find the magnitude of segments quickly.

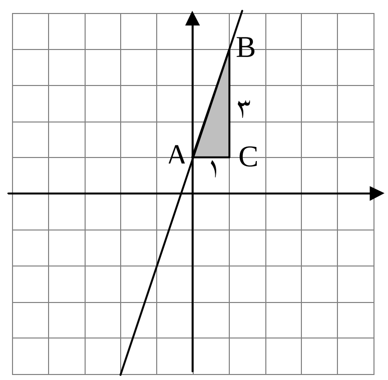
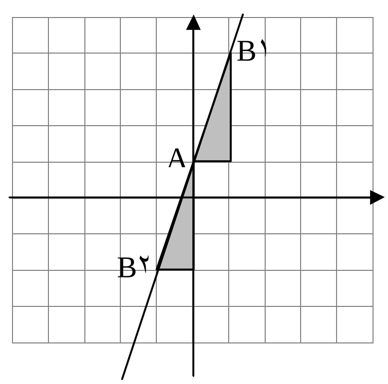


Figure ٥-٤ – sketching graph of a line

There are lots of possible points to be chosen as B which make similar triangles. Figure (٥–٥) shows two of these triangles for the given line (y = ٣x + ١).



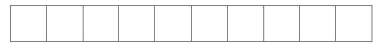


Figure ٥–٥– Similar triangles

٥-١-٦- Writing a linear equation

Another set of problems in this topic of mathematics is to write an equation for given line on the Cartesian plane. The grid lines on the plane may help students to find coordinates of two points precisely. Students typically use algebraic method to find the value of the m and n or through finding the pattern or relation between x and y values identify the corresponding linear equation. The desired equation has the form y = mx + n and students have to find the values of parameters m and n. Moschkovich et al., (١٩٩٣) state that there is an extensive literature (see e.q. Wagner and Kieran, ١٩٦٩) indicating that students have difficulty in dealing with variable or parameters, in this study, also, students had various mistake in recognizing variables and parameter, although they initially preferred to use algebraic method for solving problems. After getting involved with dynamic software activities, students learnt to find the value of m and n without using algebraic expression and or computation.

All participants in this study solved the third question in final interview in the following way. The value of n is ٣, because the line goes through the point A (٠, ٣) or meets the y-axis at point (٠, ٣). Students selected the point B on the line and draw a right angle triangle to find the value of m. The value of m is negative, because the line rises to the left. Figure ٥–٦ shows this solution and points A and B. Grid lines help student to count unit squares and find the length of segments, therefore it is important to select points which have integers as their coordinates. The point B has coordinates x = ٣ and y = -٢.

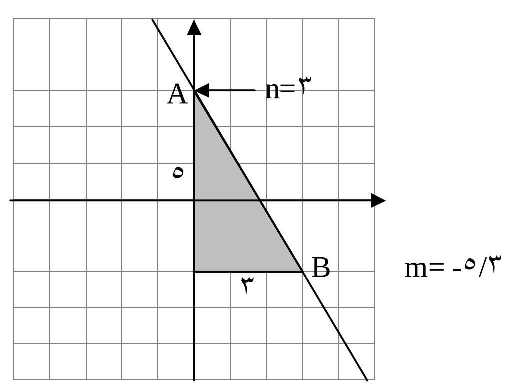


Figure ٥ – ٦ – Finding the linear equation

٥-١-٧- Issues to be considered in designing activities

One of the main goals in this study was to investigate in which ways computer based activities influence students` understanding and how designing these activities and tasks can change their knowledge and learning. Using dynamic software in this study provided opportunity for students to make a graphical sense of the nature of slope and y–intercept, but reaching this result completely depends on designing appropriate activities and tasks. Analysing data and observation of the sessions with students leads to the consideration of three major issues.

First, the dynamic software such as Geogebra provided different feedbacks for students which are faster and easier to work with independently than the ones they normally get in classes which do not use computers. Students, normally try to solve problems, and find answers, whether it is right or wrong, but when students involved in this type of activities they received automatically feedback and they had chance to correct their answers or rethink about their solutions. For instance, when I asked a student to draw a line such as y = -٣x, by selecting two points on the Cartesian plane, if the two points were chosen wrongly, student would see the equation of line corresponding to chosen points on the screen and would have a chance to see the equation immediately and find out the mistake (s)he made and do the task again. In this sense, software played tutorial role with students and made feedback to students about their conjectures which guides student to modify the answer without any biases.

Second, the most important point in designing computer based activities is how to engage students in activities and tasks. It is not enough to invite them to do simple work such as dragging and observing the change in graph or algebraic expression, because they have no motivation to focus on the data to acquire new knowledge or make a hypothesis. In this study, some activities had no enough motivation for students and they just reviewed and observed them very quickly without any considerable attention. I realised this matter when students were involved in tasks and they asked me to see previous activities again. When students returned to the activities, they had a clear question in mind to go over, thus they focused on screen to find appropriate answer for it. This interaction indicated that they gain much from manipulating with software when they have a purpose for doing that. As Healy et al., (٢٠٠٥) suggests, students gain much from interacting with software when they are involved in their own choice of project, in this study, two students asked me to make their own project by drawing lines and segments and I learnt that an appropriate task is the one that at the same time that it directs students to acquire the aims which it is designed for, is open for students to modify it based on their own interest or purpose.

Third, it is clear that all technologies have limitation for users and it influences designing computer based activities. One of the key issues here is that software's limited precision based on hardware specification such as screen resolution. In some activities, when students used the grid lines on the Cartesian plane, they were not able to choose any point they wanted, since step was already defined. Also, in one activity, software computed the value of slope and rounded the answer. This answer, sometimes, was approximately and there is a difference between this approximated value and calculated one. This problem posed some questions for careful students.

~ - - *Conclusion*

This case study attempted to identify in what ways students think about different aspects of a particular mathematical concept and how designing appropriate dynamic environment based activities and tasks influence students` learning and change their understanding of the nature of the slope and y–intercept. Five participating students from ۸th and ۹th grade initially were interviewed and answered three questions and they were invited to manipulate some computer based activities. Finally, they were answered three questions again to show a possible change in their understanding of the meaning of the slope and y–intercept of the linear equations.

The data were analysed in three levels of knowledge and students' solutions and responses were interpreted based on two dimensional framework; the major goal in using the framework for understanding functions and particularly the linear equations was to interpret students' answers and solutions for solving three questions before and after interacting with dynamic geometry software, Geogebra. There were two ways of viewing functions (the process and object perspectives) and the three representations of functions (tabular, graphical, and algebraic). The analysing data indicated that, appropriate computer based activities facilitated students to move fluently across these representations and perspectives and able to see the slope of line and y–intercept as object, although developing such flexibility is difficult and needs more research.

Researches should be planned in order to complete the framework and appropriate tasks should be designed to show students movement from one cell of the framework to another one or across different representations in a perspective or from one perspective to another one. The results of these researches could be essential for developing or revising curriculum and assessing student learning (Moschkoric et al., ۱۹۹۳).

This study tried to show the place of instructional dynamic software in the process of learning. I believe dynamic environment can assist students to change their understanding and offer them the opportunity to deal with some aspects of function and linear equation in particular, and develop students' intuitions, in ways that were simply inaccessible prior to the availability of computer based tools. Specially, appropriate tasks and activities in dynamic software allow students to manipulate linear equation and its graph as objects, and facilitate the development of object perspective in ways not possible by blackboard or paper and pencil (Moschkovich et al., ۱۹۹۳). This study described the computer based activities intended to help students to make a geometrical and graphical sense of the slope of a line and its y–intercept. Students learn to have a graphical interpretation of the slope and use the graphical method in addition to algebraic methods such as use of tables for graphing lines or writing an equation for given line. There are some additional benefits of using technology such as saving time, for learning process in terms of understanding and researchers to see some aspects of cognition at a fine–grain size (Schoenfeld, ۱۹۹۱).

˜ -˜ - *Implication for teachers and curriculum developers*

Today, the Iranian government tend to use ICTs in all possible situations such as electronic government, banking systems, e-shopping and education as well. The government has recently equipped middle and high schools with the required hardware to develop use of computers in the classroom in all subjects. Computers now are available in many homes and their facilities are accessible to students. In Ministry of Education, the main aim is to improve the standard quality of education in schools and enhance learning experiences and to supply necessary materials, and particularly to produce and apply educational software. At present, mathematical software was not widely used and there are several reasons for this incident. First, the lack of teachers` knowledge and skills of working computers and how to use it for enhancing the teaching-learning process, as well. Second, Current curriculum needs to be modified to integrate use of ICT.

This study could help curriculum developer to rethink about the impact of technology on designing mathematic curriculum and textbooks in Iran and aid them to make informed decisions about incorporating technology into the process of teaching-learning mathematics. Activities and tasks used in this study could be used by local schools to initiate and improve the use of such software in mathematics classrooms. Specifically it could assist teachers to know students' misunderstandings in the topic of study and how use of dynamic software in their teaching could make students to understand mathematics better. It is further hoped that this study will provide a good start for making Iranian mathematics educators' community to do research on the use of mathematical software.

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Appendixes

Appendix١- Permission letter for interviewing students

Appendix ٢- List of figures

Leading education and social research
Institute of Education
University of London

MA Dissertation in Mathematics Education
٠١/٠٤/٢٠٠٩ – ٠١/٠٩/٢٠٠٩

Dear Parents,

I want to inform you that I am a student at IOE and my research project for MA dissertation is about a mathematical topic (linear equation) and how dynamic software influence in students learning and understanding. In This case study I will interview students (audio recorded) and ask them to engage in using computer and doing some activities about linear equation. I will observe students and take note on how they do these tasks. All interviews and computer work session will take place at school in May. I will use no real name in my report and destroy all personal documents and tape after receiving my MA grade. I also will briefly inform students about research results after analysing data.
Your child has accepted to participate in this study. If you agree that he/ she join to this research, please, kindly sign this permission letter.

Best wishes
Khosrow Davoodi

I have read the information (above) about the research. ☐ (please tick)

I will allow the researchers to observe me/my child ☐ (please tick)

I agree to be interviewed ☐ (please tick)

Parent’s Name ______________________________

Signed ______________________________ date __________________

Student`s Name

Researcher’s name: Khosrow Davoodi

Signed date: ٢٥/٠٤/٢٠٠٩

List of figures